RSCS: Minimum measurement MMV Deterministic Compressed Sensing based on Complex Reed Solomon Coding

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***Abstract*—Compressed Sensing (CS) is an emerging field in mathematics that is used to measure few measurements of sparse vectors for lossless reconstruction. In this paper we use results from channel coding to create the recovery algorithm RSCS for CS in the Multiple Measurement Vector case (MMV) that can be used with a deterministic measurement matrix by using error correction schemes. In particular, we show that a modified Reed Solomon encoding-decoding structure can be used to measure sparsely representable vector systems down to the theoretical minimum number of measurements with guaranteed reconstruction, even in the low dimensional case.**

***Index Terms*—Compressed Sensing, Reed Solomon, MMV, Deterministic**

I. Motivation

There has been considerable interest in the emerging field of Compressed Sensing (CS), especially in the Multiple Measurement Vector (MMV) case. The main problem of CS is to reconstruct sparsely representable vector systems out of few measurements, described by a measurement matrix [1]. One of the main problems in CS theory is the design of good measurement matrices. Normally sub Gaussian random matrices are used that are well known for good recovery in high dimensions. Sadly in many applications this is not feasible to implement or the dimension of the problem is too low for random number generators to work properly. A potential solution are deterministic measurement matrices. They are easier to implement but are known so far for decreased measurement efficiency [2].

This work was motivated by the search for optimal deterministic matrices and a corresponding reconstruction algorithm that is easy to implement and still has good reconstruction abilities. Bosserts work on combining channel coding theory with Compressed Sensing [3] inspired us to take the same perspective and use this field for solving the CS problem. Especially Complex Reed Solomon (RS) codes are well known for their high error correcting capabilities [4]. The fact that RS codes can decode **sparse** error vectors can be utilized to conquer the CS setting. Sadly this only works for vectors, that directly possess a sparse number of nonzero elements.

To improve upon the results in [3], we go one step further by extending this idea to **sparsely representable** vectors, that can be sparse in any arbitrary basis.

A. Main Result

Our main result is the derivation of the CS reconstruction algorithm RSCS with a corresponding **deterministic** measurement matrix, that is inspired by Complex Reed Solomon decoding. With $\mathbf{X} = \mathbf{\Psi}\mathbf{C} \in \mathbb{C}^{N \times L}$ consisting of L vectors with N entries each and $\mathbf{C}$ being k-row sparse with no requirements towards $\mathbf{\Psi}$ other than a linear independence of the columns of $\mathbf{\Psi}$ this algorithm is **guaranteed** to retrieve $\mathbf{X}$ with the theoretical minimum of $L(k + \lceil L/k \rceil)$ measurements, even in the low dimensional case. The number of measurements is independent of $\mathbf{\Psi}$ and of the length of $\mathbf{X}$, which makes it increasingly effective for very sparse vectors. Furthermore, an estimator for the unknown value k makes the algorithm easy and efficient to use. In the following we assume without loss of generality $\mathbf{\Psi}$ to be the identity matrix $\mathbf{I}$ to ease the presentation. In other words $\mathbf{X} = \mathbf{C}$ is directly sparse in the canonical basis.

II. Combination of Interleaved Complex Reed Solomon Codes and MMV CS

Complex Reed Solomon (RS-) codes belong to the family of cyclic block codes [5]. If only some of the complex symbols get corrupted, they can be restored, regardless of the actual complex number of each incorrect symbol. To combine Compressed Sensing with Complex Reed Solomon codes, we interpret the sparse vector system $\mathbf{X}$ as the systematic part and the measurements as the parity part of the code. This way, each complex number in our original sparse vector $\mathbf{X}$ is one RS-symbol. Interleaved Complex RS codes are well known to correct Lk symbol errors with $L(k + \lceil L/k \rceil)$ parity symbols if the positions of the erroneous symbols are unknown with the help of an error locator polynomial (ELP).

Figure 1 illustrates the connection of RS decoding and CS reconstruction. The upper part shows a CS interpretation of the problem. A sparse vector $\mathbf{X}$ is measured by a matrix $\mathbf{\Phi}$ to obtain $L(k + \lceil L/k \rceil)$ measurements $\mathbf{P}$ which are sufficient to reconstruct $\mathbf{X}$ even with noise. The lower part describes the RS view. RS decoders with $L(k + \lceil L/k \rceil)$ parity symbols can correct up to Lk errors at any k positions and amplitudes, so especially the total erasure $\mathbf{E} = -\mathbf{X}$ is a correctable error, if $\mathbf{X}$ is **row sparse**. In other words, the CS algorithm can measure the $L(k + \lceil L/k \rceil)$ parity symbols of a systematic Reed

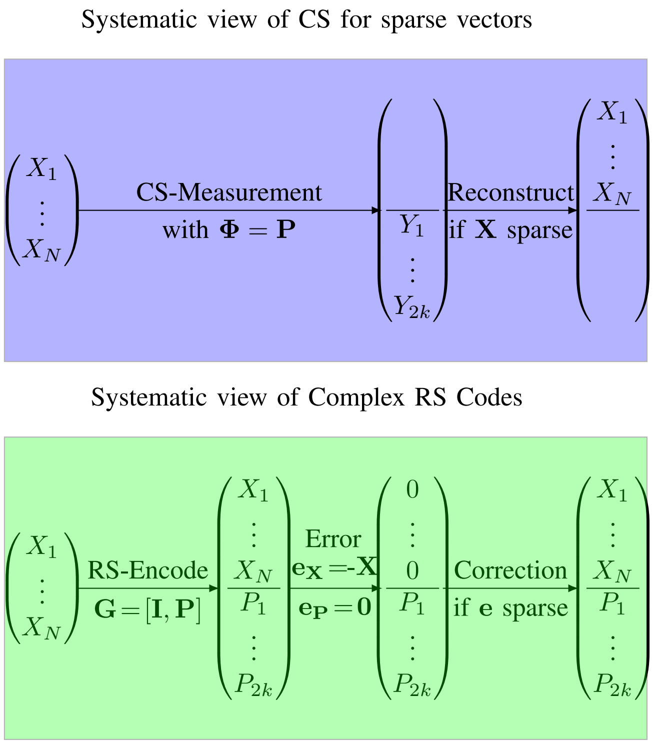

Fig. 1. Schematic for measurement and reconstruction of sparse vectors for $\mathbf{\Psi} = \mathbf{I}$ and L=1

Solomon encoding scheme and **guarantee** the recovery, if $\mathbf{C}$ is k-row sparse. The number of parity symbols is independent of the length of the original vector system $\mathbf{X}$, so its well suited for long vectors with $k \ll N$. To really combine the two theories, one crucial detail needs to be clarified. Compressed Sensing includes the vast group of sparsely representable vectors of the form

$$\mathbf{X} = \mathbf{\Psi}\mathbf{C},\ \mathbf{C} \text{ sparse} \tag{1}$$

In general, there are not exactly k nonzero elements but k columns of $\mathbf{\Psi}$ are active, e.g. k frequencies. For the algorithm to still work in this more general case, the RS decoder needs to be extended. With α as an element of order N and $\mathbf{A}_\alpha = \text{diag}(\alpha, \alpha^2, \ldots, \alpha^N)$ the error locator polynomial (ELP) is

$$\Lambda(t) = \prod_{i=1}^{k}(t - \alpha^{\mathcal{I}(i)}) = t^k + \Lambda_1 t^{k-1} + \Lambda_2 t^{k-2} + \cdots + \Lambda_k \tag{2}$$

with $\Lambda(\alpha^{\mathcal{I}(i)}) = 0$ as roots for the active support of $\mathbf{X}$. This ELP needs to be generalized to

$$\Lambda_{\mathbf{\Psi}}(\mathbf{t}) = \mathbf{1}\left(1 \cdot \mathbf{A}_\alpha^{k+1} + \Lambda_1 \mathbf{A}_\alpha^{k} + \ldots + \Lambda_k \mathbf{A}_\alpha\right)\mathbf{t} \tag{3}$$

with $\Lambda_{\mathbf{\Psi}}(\mathbf{\Psi}_{\mathcal{I}(i)}) = 0$ as roots for the active support in the sparse representation $\mathbf{\Psi}$. This way, the same theoretical boundaries that hold for $\mathbf{\Psi} = \mathbf{E}$ can be achieved for arbitrary $\mathbf{\Psi}$. The full paper will provide full details on the proof of the RSCS algorithm.

III. Comparison to Other CS algorithms

To evaluate our new proposition, we compare it to existing MMV algorithms, namely SOMP [6] and MUSIC [7]. Figure 2 shows results for the three different algorithms. The simulation was run for $N = 100$, $k = 10$, $m = 20$ and $L = 10$. As a measure of performance we depict the mean number of correctly estimated support entries averaged over 1000 Monte Carlo trials. As one can see, our Reed Solomon approach (RSCS) performs best. If the SNR is above 10dB, the reconstruction is error free every time. The MUSIC algorithm also achieves perfect reconstruction, but needs a minimal SNR of 20 dB to guarantee it. The SOMP algorithm can not guarantee reconstruction. Even in the noiseless case, only 70% of the indices can be reconstructed in the mean. So even in this mild setting and with optimized tuning parameters, SOMP fails.

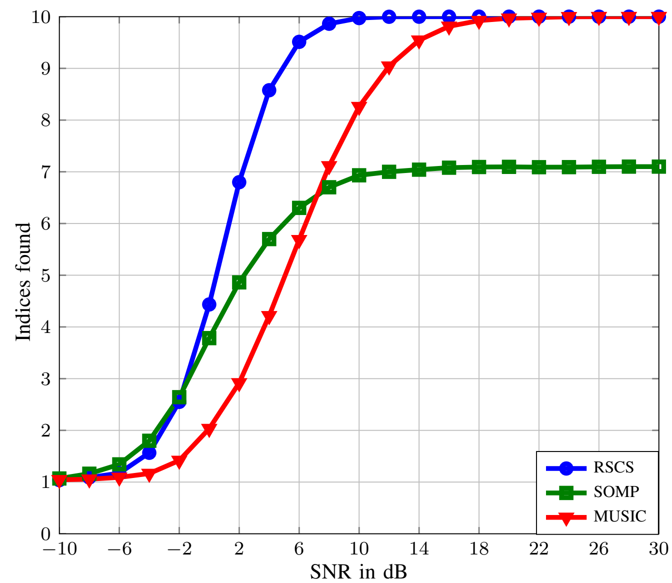

Fig. 2. Mean number of the reconstructed support indices, compared for RSCS (our contribution), SOMP and MUSIC in the setting $N = 100$, $k = 10$, $m = 20$, $L = 10$

The full paper will provide a wider range of numerical results as well as an extended bibliography.

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