Compressed Edge Spectrum Sensing for Wideband Cognitive Radios

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Abstract—Free licensed spectral bands have become rare due to the increasing number of wireless users and their demand for high data rates. Likewise, the static allocation of these bands results in an under-utilization of the spectrum. Cognitive Radio (CR) has emerged as a promising solution to the dilemma by allowing opportunistic users to transmit in the absence of licensed users. Spectrum sensing is therefore the key component of CR and coexistence management in general. In order to detect as much transmission opportunities as possible, a large bandwidth has to be monitored which according to Shannon-Nyquist necessitates high sampling rates. For fast and accurate spectrum estimation, we propose a novel approach called Compressed Edge Spectrum Sensing (CESS) which exploits the sparsity of power spectrum edges and allows for sampling down to 6% of Nyquist without losses in the detection accuracy of occupied and unoccupied spectrum regions.

I. INTRODUCTION

The number of devices with a need for higher data rates is growing due to the transition from voice-only to multimedia-type applications [1]. The concept of Cognitive Radio [2] has evolved as a result of the ever-increasing demand for new spectral bands and the under-utilization of those already statically allocated. Herein, secondary users are allowed to use vacant bands in the absence of licensed primary users. Similar, Coexistence Management (CM) [3] for Industry 4.0 applications should enable coordination of several wireless connected systems in crowded industrial environments. Therefore, a crucial aspect of CR and CM is spectrum sensing to detect these opportunities for transmission, the so-called white spaces. In order to find as much white spaces as possible and to enhance throughput, a wideband must be sensed with the result of high Nyquist rates. Otherwise, a bank of tunable narrowband bandpass filters, to search one narrowband at a time, would become necessary, which means a considerable implementation challenge and results in a high power consumption not favored in wireless devices. Fortunately, several measurements have shown the spectrum to be under-utilized [1] so that it can be assumed sparse. Therefore, Compressed Sensing (CS) is often mentioned as a way of reducing the load [4], [5]. However, two problems result. On the one hand, CS algorithms add processing complexity and delay. Since the time for detecting sudden interference to a primary user has to be as low as possible, algorithms have to be chosen that can be executed very fast. On the other hand, the PSD may not always be sparse but crowded, e.g. in a busy environment, which makes application of CS difficult and requires representation in another basis. Because many spectra in practical applications like those of OFDM transmissions exhibit sharp boundaries and can be approximated as piecewise flat, the derivative has a few non-zero elements which depict edges. Thus, it seems reasonable to use the edge spectrum as the sparsity basis which we exploit in this paper. In the following, the underlying theory is described and extended to build a whole processing chain which is able of seeking and classifying occupied and free frequency bands.

II. SYSTEM MODEL AND PROBLEM STATEMENT

Usually, the Nyquist rate equaling the bandwidth of interest is chosen and $N$ samples $x \in \mathbb{C}^{N \times 1}$ of the continuous signal $x(t)$ are obtained in order to perform spectrum sensing. The signal consists of the several transmissions $x = \sum_i x_i$, with a piecewise flat Power Spectrum Density (PSD) of height $\sigma_i^2$ as illustrated in Fig. 1. The transmissions have to be detected and are additionally supposed to be zero-mean wide-sense stationary. These assumptions are valid for many signals appearing in real world settings, i.e. OFDM transmissions [6], since well-defined spectrum masks are employed. To introduce CS, we consider a subsampling approach. It is represented by the subsampling matrix $V \in \mathbb{C}^{M \times N}$ and generates compressed measurements $y \in \mathbb{C}^{M \times 1}$. Relating the measurements to the amplitude spectrum $c$ through

$$y = Vx = VF^{-1}c = Ac$$

(1)

with $F \in \mathbb{C}^{N \times N}$ being the DFT matrix, the CS reconstruction problem

$$\hat{c} = \arg \min_c \|c\|_0 \quad \text{s.t.} \quad y = Ac$$

(2)
can be formulated to find \(c\) with under-determined \(A\) assuming sparsity in \(e\).

### A. Power Spectrum Sensing

To reconstruct the PSD instead of the amplitude spectrum, we make use of the autocorrelation matrix [5]:

\[
R_y = \mathbb{E}[y y^H] = \mathbb{E}[A c (A c)^H] = A R_c A^H.
\]

By definition [5] \(s = \mathbb{E}[|c|^2]\) holds so that we can find the entries of \(s\) on the diagonal of \(R_c\). Owing to the wide-sense stationarity of all signals, \(R_c\) is a diagonal matrix. After vectorization and exploitation of the diagonal structure of \(\text{vec}(R_c)\), the following relation can be obtained according to [5]:

\[
\text{vec}(R_y) = (A^* \otimes A) \text{vec}(R_c) = (A^* \otimes A) s = \Phi s.
\]

Here, \(A^*\) denotes the complex conjugate of \(A\), \(\otimes\) the Kronecker product, and \(\otimes\) the column-wise Kronecker product, also known as "Khatri-Rao" product. The new PSD sensing matrix \(\Phi\) has the dimensions \(M^2 \times N\) and (4) may thus exhibit a unique solution for \(M^2 \geq N\). However, the minimum number of measurements required to solve (4) uniquely is derived in [5] and equals \(M > N/2\) due to the specific problem structure. In fact, this allows for subsampling without assuming sparsity and solving the overdetermined equation system where \(r_y = \text{vec}(R_y)\) with the least squares (LS) method:

\[
\hat{s} = \arg \min_s ||r_y - \Phi s||_2 \Rightarrow \hat{s} = \Phi^d r_y.
\]

\(\dagger\) indicates the Moore-Penrose pseudoinverse and enables solutions also for underdetermined equation systems which can be treated as \(l_2\)-approximations of the \(l_0\)-norm in the equivalent CS reconstruction problem. This problem can be stated as

\[
\hat{s} = \arg \min_s ||s||_0 \text{ s.t. } r_y = \Phi s
\]

and allows for further subsampling since it can be solved uniquely for \(M > ||s||_0\) as derived in [5]. In other words, a minimum average sampling rate equaling the actual occupied bandwidth becomes necessary lowering the requirements in a practical implementation. The compression ratio reads \(\kappa = M/N > ||s||_0/N\). This means a reduction of 50% compared to (2) and is due to the assumptions made about stationarity. For the previous bounds to be valid, \(A\) has to be full spark.

### B. Practical considerations

In a practical implementation, solving (6) requires \(Q\) measurements \(y_i\) to approximate the expected value of the autocorrelation by the mean where high values of \(Q\) lead to a high noise or error reduction:

\[
\hat{R}_y = \frac{1}{Q} \sum_{i=1}^{Q} y_i y_i^H.
\]

These measurements are obtained in a time window of length \(\Delta t\) assuming wide-sense stationarity as illustrated in fig. 2. It shows the varying occupation of spectrum which remains constant in one time window. If \(T_s\) denotes the Nyquist sampling period and \(N\) the frequency resolution, the whole sampling time for (7) amounts to \(\Delta t = Q N T_s\). Therefore, a trade-off between fast detection, resolution and noise suppression has to be considered. Because noise is present in practical settings, (6) has to be modified to include a bounded error \(\epsilon\) of the \(l_2\)-norm:

\[
\hat{s} = \arg \min_s ||s||_0 \text{ s.t. } ||r_y - \Phi s||_2 \leq \epsilon.
\]

### III. Compressed Edge Spectrum Sensing

#### A. One-dimensional Edge Spectrum Sensing

Unfortunately, there is no guarantee for the spectrum to be sparse since it can be crowded or full in worst case. One solution to overcome this problem is to consider edges \(z\) of the piecewise constant spectrum due to the fact that there are considerably fewer edges (exactly \(J\)) than occupied entries of the PSD and entries in general (\(J = ||z||_0 \leq ||s||_0 \ll N\)). The definition of a new CS problem where the power spectrum in (4) is replaced by the edge spectrum immediately suggests itself. Both are related to each other through a numerical derivative

\[
s = \Gamma^{-1} z,
\]

with \(\Gamma \in \mathbb{C}^{N \times N}\) being the difference matrix

\[
\Gamma = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
-1 & 1 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & -1 & 1
\end{bmatrix}.
\]

This leads to the CS problem

\[
\hat{z} = \arg \min_z ||z||_0 \text{ s.t. } r_y = (\Phi \Gamma^{-1}) z
\]

defining a new approach that combines the ideas of [4] and [5]. In the following, we designate this approach one-dimensional Compressed Edge Spectrum Sensing (1D CESS). It allows unique reconstruction of the edge spectrum for compression ratios up to \(\kappa = M/N > J/N\) which is a result of the fact that the new sensing matrix \(\Phi \Gamma^{-1}\) has the same spark as the old one. Just the mapping to the null space changes,
B. Two-dimensional Edge Spectrum Sensing

So far, only edges in the frequency domain have been exploited. But the spectrum also exhibits edges in the time domain as depicted in Fig. 2. This stems from the fact that communication systems are only active for a limited time to adhere to regulation or because of intermittent activity. Hence, this additional structure can be used to create an advanced spectrum sensing algorithm based on the Total Variation Norm in two dimensions. For this purpose, $K$ equations from the previous section obtained in time windows of length $\Delta t$ with subsampling matrices $\Phi$ are stacked together into one equation system:

$$
\mathbf{r}_T = 
\begin{bmatrix}
\mathbf{r}_{y_1} \\
\mathbf{r}_{y_2} \\
\vdots \\
\mathbf{r}_{y_K}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{\Phi}_1 & 0 & \cdots & 0 \\
0 & \mathbf{\Phi}_2 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \mathbf{\Phi}_K
\end{bmatrix} 
\begin{bmatrix}
\mathbf{s}_1 \\
\mathbf{s}_2 \\
\vdots \\
\mathbf{s}_K
\end{bmatrix} = \mathbf{\Phi}_T \cdot \mathbf{s}_T. 
$$

(12)

For application of CS, the sampling process in a practical system should be designed to include randomness [7]. Hence, the $\Phi$ are different from each other so that additional information can be used. In order to reconstruct the edges in the time and frequency domain, the 2D-differential of $\mathbf{s}_T$ has to be minimized which leads to the following Total Variation Norm Minimization Problem:

$$
\hat{\mathbf{s}}_T = \arg \min_{\mathbf{s}_T} \| \mathbf{\Gamma}_{2D} \mathbf{s}_T \|_1 \quad \text{s.t.} \quad \mathbf{r}_T = \mathbf{\Phi}_T \hat{\mathbf{s}}_T. 
$$

(13)

The 2D-differential matrix $\mathbf{\Gamma}_{2D} = [\mathbf{\Gamma}_f, \mathbf{\Gamma}_t]^T$ consists of the differential in the frequency domain $\mathbf{\Gamma}_f = \mathbf{I}_K \otimes \mathbf{\Gamma}_1$ with $\mathbf{\Gamma}_1 \in \{-1, 0, 1\}^{N \times N}$ denoting

$$
\mathbf{\Gamma}_1 = 
\begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 1
\end{bmatrix}
$$

(14)

and the differential in the time domain $\mathbf{\Gamma}_t \in \{-1, 0, 1\}^{(K-1)N \times KN}$:

$$
\mathbf{\Gamma}_t = 
\begin{bmatrix}
-1_{1,1} & 0 & \cdots & 0 \\
0 & -1_{2,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -1_{(K-1)N,(K-1)N} \\
1_{N+1,1} & 0 & \cdots & -1_{(K-1)N,(K-1)N} \\
0 & 1_{N+2,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1_{KN,(K-1)N}
\end{bmatrix}^T.
$$

(15)

From here on, this algorithm will be denoted as 2D CESS. The motivation for applying it is simple: The necessary minimum number of measurements should intuitively decrease when considering piecewise-constant spectra as depicted in Fig. 2 since more information is gained than has to be reconstructed. In comparison to the one-dimensional case, the results are expected to allow for greater compression and performance, respectively. But this should happen at the cost of a delay $T = K \Delta t$, the time for taking the samples.

IV. PROCESSING CHAIN

Fig. 3 shows the whole proposed processing chain to perform spectrum sensing. First, the signals arrive through a mitigating channel $\mathbf{H}$ and are superimposed with AWGN. After subsampling, the power spectrum $\mathbf{s}$ is reconstructed according to (5), (6), (11) and (13), respectively. Now, the question has to be asked where the bands are located and whether they are occupied or not. Therefore, we apply a simple feature detection. It consists of the Wavelet Edge Detector (WED) and the Energy Detector (ED) [6], [1].

First, the WED detects the band boundaries $\tilde{f}_i$ by looking for the local maxima in the derivative $z$. We notice that direct reconstruction of edges with CESS should make this step redundant since boundaries are expected to be where the derivative is non-zero. However, CS reconstruction as well as spectrum shapes are not perfect and noise makes it impossible to find the true solution. In order to suppress noise induced edges, we thus additionally apply a thresholding operation. For the sake of simplicity, we do not apply a Gaussian kernel on the PSD or use it even in the reconstruction step, in contrast to [4]. In order to choose a reasonable threshold $\eta_{\text{WED}}$, its relation to detection and false alarm rates has to be derived:

To begin with, we model the amplitude spectrum as a multivariate complex gaussian random vector $\mathbf{c} \sim \mathcal{CN}(0, \Sigma)$ with covariance matrix $\Sigma$. Here, $\Sigma = \text{diag} \{ \mathbf{\Sigma}_1, \mathbf{\Sigma}_2, \ldots \} + \sigma^2 N \mathbf{I}_N$ is a block diagonal matrix where each band transmission is represented by its power level $\mathbf{\Sigma}_i = \sigma_i^2 \mathbf{I}_{B_i}$ and bandwidth $B_i$. Some bands are empty ($\sigma_i^2 = 0$) and consist only of the noise level $\sigma_n^2$. So the received Nyquist-sampled signal $\mathbf{x}$ consists of additive correlated Gaussian noise. According to (3) and (7), the average of the squared absolute values of $\mathbf{c}$ is the approximated estimate of the power spectrum. For $Q = 1$ one point is distributed according to the Chi-square distribution $\chi^2$ with two degrees of freedom and variance $(\sigma_1^2 + \sigma_n^2)^2$. If more realizations are examined, the variance decreases by the number of frames $Q$. Additionally, this results in convergence to a normal distribution according to the central limit theorem. After numerical derivation, the variance $\text{Var}[Z]$ of one point in the edge spectrum $z$ is the sum of two neighboring variances in the PSD. Hence, the rate for exceeding an absolute value $\eta_{\text{WED}}$

$$
P_{\text{F, WED}} = 2 \cdot \Phi \left( -\eta_{\text{WED}} / \sqrt{\text{Var}[Z]} \right), 
$$

(16)

can be stated, with $\Phi$ denoting the cumulative function of the standard normal distribution. In order to generate a threshold satisfying a false alarm rate $P_{\text{F, WED}}$, we invert the equation.
We choose the part of the spectrum with highest signal power \( \sigma^2 \), for calculating the variance \( \text{Var}[Z] = 2 (\sigma^2 + \sigma_n^2)^2 / Q \) of the random variable since we have to cover the worst case. Analogously, the missed detection rate

\[
P_{\text{MD,WED}} = \Phi \left( \frac{\eta_{\text{WED}} - \mu}{\sqrt{\text{Var}[Z]}} \right) - \Phi \left( \frac{-\eta_{\text{WED}} - \mu}{\sqrt{\text{Var}[Z]}} \right) \quad (17)
\]

at one noise-signal edge with variance \( \text{Var}[Z] = \sigma^2 / Q + (\sigma^2 + \sigma_n^2)^2 / Q \) and height \( \mu = \sigma^2 - \sigma_n^2 \) can be derived.

In the last step, the Energy Detector similarly compares the average carrier amplitude between two boundaries to a threshold depending on the noise level. In contrast to literature [1], [8], we choose it to be the same except for the noise threshold depending on the noise level. In contrast to literature, the average carrier amplitude between two boundaries to a threshold depending on the noise level. In contrast to literature, the average carrier amplitude between two boundaries to a threshold depending on the noise level.

We modulated them to a randomly selected carrier and fixed the occupation \( \beta = \|s\|_0 / N \) to maintain the same basic conditions. For investigation of 2D CESS, also time behavior had to be modeled. Hence, a Markov model with the states empty and occupied was defined for every single band with the same mean and starting occupation \( \beta \).

Here, the probability for changing the state after one time slot \( \Delta t \) from empty to occupied was set to 1/30 and 1/20 vice versa. For CS reconstruction, we used the OMP (Orthogonal Matching Pursuit) algorithm, and for TVN Minimization, the convex optimization toolbox CVX [9] with the solver SDPT3. In both cases, we chose the stopping criterion to be the true residual \( \epsilon = \| \mathbf{R}_y - \hat{\mathbf{R}}_y \|_F \) calculated with the ideal spectrum shape in order to show the best theoretical performance. In practice, other criteria have to be used. Table I shows the default simulation parameters reflecting practical settings. An appropriate threshold \( \eta_{\text{WED}} \) to realize a sufficient dynamic range for the test signals can be obtained by dynamically estimating \( \text{Var}[Z] = 2 \cdot \max(s)^2 / Q \) and choosing \( P_{\text{WED}} = 0.1\% \). To fulfill Null Space Criterion and Restricted Isometry Property with high probability, the subsampling matrix \( \mathbf{V} \) was set to a Gaussian random matrix with normalized columns. Finally, the number of Monte Carlo Trials was chosen as 1000.

\[ \text{TABLE I} \]

<table>
<thead>
<tr>
<th>parameter</th>
<th>( N )</th>
<th>( Q )</th>
<th>( \kappa )</th>
<th>( \beta )</th>
<th>( \sigma^2 )</th>
<th>( P_{\text{MD,WED}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1000</td>
<td>0.06</td>
<td>0.4</td>
<td>0.1</td>
<td>0.1%</td>
<td></td>
</tr>
</tbody>
</table>

In the last step, the Energy Detector similarly compares the average carrier amplitude between two boundaries to a threshold depending on the noise level.

**B. 1D CESS**

In Fig. 4 the detection rates \( P_D \) and the false alarm rates \( P_F \) for different methods are illustrated as a function of the compression \( \kappa \). Above \( \kappa = 15\% \), LS, CS and CESS all show perfect reconstruction. Below 15\%, LS has a higher detection rate than CS, but vastly more false alarms. In contrast, CESS begins to deteriorate significantly at a very low compression ratio of 6\%. In the chosen test setup a total of \( J = 5 \) edges in contrast to 40 spectral points have to be reconstructed. Therefore, the reconstruction quality is adequate even at the minimal compression \( \kappa = 6/100 > J/N = 5/100 \) which offers unique reconstruction. Altogether, it can be clearly seen that CESS outperforms the other two approaches in the test.
However, 2D CESS introduces a delay due to the longer time the additional structural information in the time domain.

In summary, it can be stated that in accordance with the first guess 2D CESS offers performance benefits because it utilizes the additional structural information in the time domain. However, 2D CESS introduces a delay due to the longer time window $T = K \Delta t$. For a delay oriented comparison, we fixed the threshold to $\eta_{\text{WED}} = 1.5$ and depicted $Q$ in fig. 6. We can see that at lower $Q$ mainly false alarm performance is better for 2D CESS. As a result, $Q$ can be reduced i.e. by a factor of $K = 20$ from 2000 to 100 for constant $P_k \approx 5\%$ in comparison to 1D CESS. Hence, both 2D and 1D CESS can have the same sampling time at a comparable false alarm rate. But this does not hold for detection rates: A reduction by $K \approx 6.7$ from 2000 to 300 is possible here. In conclusion, 2D CESS introduces a delay which extends even further because

### Table II

<table>
<thead>
<tr>
<th>$\kappa$ [%]</th>
<th>3</th>
<th>6</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D CESS [dB]</td>
<td>-5.27</td>
<td>-12.57</td>
<td>-18.49</td>
<td>-27.44</td>
</tr>
</tbody>
</table>

setup. The OMP needs less iterations, so CESS is also a lot faster compared to classic CS regardless of occupation.

### C. 2D CESS

The performance of the spectral estimator in terms of detection and false alarm rates when using 2D CESS and $K = 20$ can be deduced from Fig. 5. In comparison to 1D CESS, an increase in performance can be observed: Slightly higher detection rates can be achieved for relevant compression up to 3% whereas false alarms can be significantly reduced at least by half, i.e. from 7.5% to 1% at $\kappa = 6\%$. Another measure for comparing the performance regarding reconstruction accuracy is the normalized Mean Square Error (nMSE) which is related to the ideal spectrum and given for some values of $\kappa$ in table II. It can be clearly seen that 2D CESS achieves an at least 3 dB smaller nMSE for relevant compression than 1D CESS.

In summary, it can be stated that in accordance with the first guess 2D CESS offers performance benefits because it utilizes the additional structural information in the time domain. However, 2D CESS introduces a delay due to the longer time window $T = K \Delta t$. For a delay oriented comparison, we fixed the threshold to $\eta_{\text{WED}} = 1.5$ and depicted $Q$ in fig. 6. We can see that at lower $Q$ mainly false alarm performance is better for 2D CESS. As a result, $Q$ can be reduced i.e. by a factor of $K = 20$ from 2000 to 100 for constant $P_k \approx 5\%$ in comparison to 1D CESS. Hence, both 2D and 1D CESS can have the same sampling time at a comparable false alarm rate. But this does not hold for detection rates: A reduction by $K \approx 6.7$ from 2000 to 300 is possible here. In conclusion, 2D CESS introduces a delay which extends even further because $l_1$-optimization has a higher computational complexity than OMP. A detailed complexity analysis is left for future work.

### VI. CONCLUSION

The main results of called work can be summarized as follows: CESS can provide a lower compression ratio than the CS and LS approach due to exploitation of the inherent signal structure. Good performance can be achieved until the actual edge spectrum occupation of 6% is reached. 2D CESS allows for an even better performance if time-domain edges are exploited.

### ACKNOWLEDGMENT

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