Advanced Topics in Digital Communications
Spezielle Methoden der digitalen Datenübertragung

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Lecture
Thursday, 10:00 – 12:00 in N3130

Exercise
Wednesday, 14:00 – 16:00 in N1250
Dates for exercises will be announced during lectures.

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Outline

Part 1: Linear Algebra
- Eigenvalues and eigenvectors, pseudo inverse
- Decompositions (QR, unitary matrices, singular value, Cholesky)

Part 2: Basics and Preliminaries
- Motivating systems with **Multiple Inputs and Multiple Outputs** (multiple access techniques)
- General classification and description of MIMO systems (SIMO, MISO, MIMO)
- Mobile Radio Channel

Part 3: Information Theory for MIMO Systems
- Repetition of IT basics, channel capacity for SISO AWGN channel
- Extension to SISO fading channels
- Generalization for the MIMO case

Part 4: Multiple Antenna Systems
- SIMO: diversity gain, beamforming at receiver
- MISO: space-time coding, beamforming at transmitter
- MIMO: BLAST with detection strategies
- Influence of channel (correlation)

Part 5: Relaying Systems
- Basic relaying structures
- Relaying protocols and exemplary configurations
Outline

- Part 6: In Network Processing
- Part 7: Compressive Sensing
  - Motivating Sampling below Nyquist
  - Reconstruction principles and algorithms
  - Applications
Multiple Antenna Systems

- Exploiting Multiple Antennas for Diversity Enhancement
- SIMO
  - Diversity, Maximum Ratio Combining (beam forming at receiver)
- MISO
  - Beam forming at transmitter
  - Space-Time Coding
    - Orthogonal Space-Time Blockcodes
    - Space-Time Trellis Codes
- MIMO: Layered Space-Time Codes (BLAST) with detection strategies
  - Maximum-Likelihood, Linear Equalization
  - V-BLAST detection algorithm
  - QR-based Successive Interference Cancellation, SQRD
  - Sphere Detection
Exploiting Multiple Antennas
for Diversity Enhancement
Motivation for Antenna Diversity (1)

- Flat Rayleigh fading channel
  \[ y[k] = h[k] \cdot x[k] + n[k] \]

- Statistic of channel coefficient \((\sigma_h^2 = 1)\)
  - Magnitude is Rayleigh distributed
  - Squared magnitude is chi-squared distributed with 2 degrees of freedom

- Bit Error Rate
  - BER is random variable depending on \(|h|^2\)
  - Average (ergodic) BER
    \[
    \overline{P_b} = \mathbb{E}\{P_b(|h|^2)\} = \int_0^\infty P_b(|h|^2 = \xi)p_{|h|^2}(\xi)d\xi = \frac{1}{2}\left[1 - \sqrt{1 - \frac{1}{1 + \frac{E_b}{N_0}}}\right]
    \]

- Outage probability for a certain target error rate
  \[
  P_{\text{out}}(P_{b,\text{target}}) = P_r\{P_b > P_{b,\text{target}}\} = 1 - \exp\left(-\frac{[E_b/N_0]_{\text{target}}}{E_b/N_0}\right)
  \]
Motivation for Antenna Diversity (2)

- Outage probability for Rayleigh fading channel (for BPSK transmission)

- Utilization of diversity to increase performance of wireless communication

\[ P_{out} \]

\[ E_s / N_0 \text{ in dB} \]

\[ P_t = 0.25 \]
\[ P_t = 10^{-1} \]
\[ P_t = 10^{-2} \]
\[ P_t = 10^{-3} \]
\[ P_t = 10^{-4} \]

\( \rightarrow \) ergodic
What is Diversity?

- Different sources of diversity: Frequency, Time, Polarization, Code, Space
- General: receive $D$ statistically independent replicas of same signal
  - **Maximum Ratio Combining (MRC)** represents maximum likelihood estimation
    \[
    \tilde{x} = \sum_{j=1}^{D} h_j^* \cdot y_j = \sum_{j=1}^{D} h_j^* \cdot (h_j x + n_j) \\
    = x \cdot \sum_{j=1}^{D} |h_j|^2 + \sum_{j=1}^{D} h_j^* n_j
    \]
  - BER analysis
    - Receive power at each branch $E_s|h_j|^4$
    - Noise term contains sum of $D$ i.i.d. Gaussian processes, weighted by $h_j^*$  
      $\rightarrow$ zero-mean Gaussian process with variance $\alpha_n^2 \sum_{j=1}^{D} |h_j|^2$
  - Average receive power per Bit after MRC: $E_s \left( \sum_{j=1}^{D} |h_j|^2 \right)$
SNR Distribution for Maximum Ratio Combining

- MRC: constructive superposition of independent signal parts
- Equivalent SISO channel

![Diagram showing signal processing](image)

\[ P_b = \frac{1}{2} \operatorname{erfc} \left( \sqrt{\frac{E_s \left( \sum_{j=1}^{D} |h_j|^2 \right)^2}{N_0 \cdot \sum_{j=1}^{D} |h_j|^2}} \right) = \frac{1}{2} \operatorname{erfc} \left( \sqrt{\sum_{j=1}^{D} |h_j|^2 \frac{E_s}{N_0}} \right) = \frac{1}{2} \operatorname{erfc} \left( \sqrt{\sum_{j=1}^{D} \gamma_j} \right) \]

- Distribution of signal to noise ratio after maximum ratio combining
  - Chi-squared distribution with \(2D\) degrees of freedom

\[ p_\gamma(\xi) = \frac{\xi^{D-1}}{(D-1)! \cdot (E_s/N_0)^D} \cdot \exp \left( -\frac{\xi}{E_s/N_0} \right) \]
SNR distribution and BER for Maximum Ratio Combining

- Density approaches Dirac impulse for $D \to \infty \to \text{AWGN}$
  - Error rate performance reaches AWGN channel for $D \to \infty$
  - **Slope of curve** increases for growing $D$

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![Graph showing SNR distribution and BER for Maximum Ratio Combining](image)

**Part 4: Space-Time Signal Processing**
Single-Input Multiple-Output Systems (SIMO)

- Multiple antennas only at receiver
  \[ y = h \cdot x + n \]

- Optimal receiver performs spatial matched filtering (Rx-beamforming)
  - **Matched filter** maximizes SNR by maximum ratio combining (MRC)
    \[
    \tilde{x} = \frac{h^H}{\|h\|} \cdot y = x \cdot \frac{1}{\|h\|} \cdot \sum_{j=1}^{N_R} |h_j|^2 + \tilde{n} = x \cdot \|h\| + \tilde{n}
    \]
Gain after Maximum Ratio Combining

- MRC transforms SIMO model into a SISO channel with maximized SNR

- Two different gains:
  - **Antenna gain** in dB: \(10 \log_{10}(N_R)\)
  - **Diversity gain** due to averaging statistically independent channels
  - Normalizing signal to noise ratio after MRC hides antenna gain for illustration of diversity effect

\[
\gamma = \frac{\text{SNR}}{N_R} = \frac{1}{N_R} \cdot \sum_{j=1}^{N_R} |h_j|^2 \cdot \frac{E_s}{N_0}
\]
MASI Measurement for IEEE802.11a (36 Mbit/s-Mode)

rx-ant 1, BER: 4.87e-001

rx-ant 2, BER: 7.46e-002

rx-ant 3, BER: 3.29e-001

rx-ant 4, BER: 2.89e-002

MRC 1+2, BER: 2.04e-002

MRC 1+2+3+4, BER: 0.00+000

BER after FEC decoding

Part 4: Space-Time Signal Processing
SNR Distribution after Maximum Ratio Combining

- Rayleigh fading channels
- i.i.d. coefficients
  - Sufficient antenna spacing required
  - Chi-squared distribution with $2N_R$ degrees of freedom

\[ p_\gamma(\xi) = \frac{\xi^{N_R-1}}{(N_R - 1)!} \cdot e^{-\xi} \]
Error Rate Performance for Diversity

- Error rate performance reaches AWGN channel for $N_R \to \infty$
- Slope of curve increases for growing $N_R$
- Diversity gain

\[ g_D \triangleq - \lim_{\gamma \to \infty} \frac{\log(P_s)}{\log(\gamma)} \]

\begin{align*}
\gamma & \approx \gamma^{-2} \\
\gamma & \approx \gamma^{-4}
\end{align*}

Part 4: Space-Time Signal Processing
Comparison of Rayleigh and Rice Fading

- Rice suffers less from fading due to line-of-sight path
- Rice channel reaches the AWGN channel with less diversity

![Graph comparing Rayleigh and Rice fading](image)

**Legend:**
- AWGN
- $D=1$
- $D=2$
- $D=4$
- $D=10$
- $D=20$

- Rayleigh
- Rice ($K=10$)
Comparison of Rayleigh and Rice Fading

- No diversity gain for Rice factor $K \to \infty$ due to normalization and no fading
- Diversity concepts are only an appropriate means in severe fading conditions

$$E_s/N_0 = 12 \text{ dB}$$
Influence of Correlation between Diversity Paths

- Diversity gain vanishes for increasing correlation ($\rho \to 1$), here for BPSK with identical distributed channels at $E_s/N_0 = 12$ dB.
Transmit Diversity?

- Receive diversity can be achieved with multiple receive antennas.
- Can transmit diversity be obtained by transmitting the same signal with \( N_T \) antennas?

\[
y = \frac{x}{\sqrt{N_T}} \cdot \sum_{i=1}^{N_T} h_i + n
\]

- Average receive power per symbol

- Coefficients \( h_i \) are i.i.d. with variance \( \sigma_h^2 = 1 \)
- New Rayleigh distributed coefficient with \( \sigma_h^2 = N_T \)

- Error probability

\[
P_b = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_s}{N_T \cdot N_0}} \right) = \frac{1}{2} \text{erfc} \left( \sum_{i=1}^{N_T} h_i \sqrt{\frac{E_s}{N_T \cdot N_0}} \right)
\]
Multiple-Input Single-Output Systems (MISO)

- Multiple antennas only at transmitter
  - Appropriate pre-processing required

- Different levels of channel knowledge (channel state information, CSI) at transmitter
  - Perfect channel knowledge $\rightarrow$ beam forming
  - No channel knowledge $\rightarrow$ space-time coding
  - Total transmit energy normalized to $E_s = E\{||x||^2\}$

\[
y = h \cdot x + n = h \cdot As + n
\]

\[
h = [h_1 \ h_2 \ \cdots \ h_{NT}]
\]
MISO Systems: Perfect Channel Knowledge at Transmitter

- Optimum transmitter maximizes SNR at receiver by using matched filter with normalized transmit power: \( A = \mathbf{h}^H / \|\mathbf{h}\| \)

\[
\tilde{s} = \mathbf{h} \cdot \frac{\mathbf{h}^H}{\|\mathbf{h}\|} s + n = s \cdot \frac{1}{\|\mathbf{h}\|} \sum_{i=1}^{N_T} |h_i|^2 + n
\]

\[= \|\mathbf{h}\| \cdot s + n\]

- **Tx-Beamforming** by maximum ratio combining

- Two different gains
  - Antenna gain: \(10 \log_{10}(N_T)\) in dB
  - Diversity gain

- MISO system with perfect CSI at transmitter equivalent to SIMO system
Extension to MIMO-Systems: Multilayer-Transmission

- Assuming instantaneous knowledge at transmitter and receiver
- Singular value decomposition of channel matrix $H = U \Sigma V^H$

- Exploiting all eigenmodes of channel supports multiple data streams
  - Transmitter $x = V \cdot s$
  - Receiver $\tilde{s} = U^H \cdot y = U^H \cdot (Hx + n)$
    $$\tilde{s} = U^H \cdot U \Sigma V^H \cdot V s + U^H \cdot n = \Sigma \cdot s + \tilde{n}$$
    $$\tilde{s}_i = \Sigma_i \cdot s_i + \tilde{n}_i$$

- Transforming MIMO system into parallel SISO systems by singular value decomposition
- Number of parallel layers depend on rank of $H$
- Adaptation of modulation/coding per parallel layer by water-filling
Space-Time Codes

- General principle of STC
- Error Rate Analysis of MIMO Systems
- Space-Time Blockcodes
- Space-Time Trelliscodes
Space-Time Codes (STC)

- **Space-Time Codes (STC)**
  - Achieve transmit diversity without requiring CSI@Tx
  - Coding = arranging the transmitted symbols in space and time
- **Orthogonal Space-Time Block Codes (STBC)**
- **Space-Time Trellis Codes (STTC)** also provide coding gain
- ... Transmit diversity schemes can be combined with multiple receive antennas!

- **Transmission of block of length** $L \rightarrow$ code matrix $X$

$$X = \begin{bmatrix} x[1] & x[2] & \ldots & x[L] \\ \vdots \\ x_{NT}[1] & x_{NT}[2] & \ldots & x_{NT}[L] \end{bmatrix}$$

- **Space-Time Code specifies how the code matrix $X$ is generated**
  - Mapping of information symbols $s_1, \ldots, s_m$ onto transmit symbols $x_i[k]$
  - Appropriate design criteria for STC are required!
Instantaneous Pairwise Error Probability

- Probability to decide in favor of code matrix $E$, when $X$ was transmitted
  - Squared Euclidian distance of corresponding received sequences
    $$d^2(X, E|H) = \sum_{k=1}^{L} \|H \cdot (x[k] - e[k])\|^2 = \sum_{j=1}^{N_R} h_j \cdot \Delta(X, E) \cdot h_j^H$$
  - With squared distance matrix and eigenvalue decomposition
    $$\Delta(X, E) = (X - E) \cdot (X - E)^H = U \Lambda U^H$$
    the squared Euclidian distance becomes
    $$d^2(X, E|H) = \sum_{j=1}^{N_R} h_j \cdot U \Lambda U^H \cdot h_j^H = \sum_{j=1}^{N_R} \Lambda \sum_{i=1}^{N_T} |b_{j,i}|^2 \lambda_i$$
    with $b_j = h_j \cdot U$ elements $b_{j,i}$ of $b_j$ are still Rayleigh distributed with same variance

- $P\{X \rightarrow E|H\} \propto \exp \left( -\frac{E_s}{4N_0} d^2(X, E|H) \right)$
Average Pairwise Error Probability

- **Average pairwise error probability**

\[
P\{\mathbf{X} \rightarrow \mathbf{E}\} = E_H\{P\{\mathbf{X} \rightarrow \mathbf{E}|\mathbf{H}\}\} = E_\beta \left\{ \prod_{j=1}^{N_R} \prod_{i=1}^{N_T} \exp \left( -\frac{E_s}{4N_0} |b_{j,i}|^2 \lambda_i \right) \right\}
\]

- **Calculation of expected value w.r.t to \( \beta \) yields**

\[
P\{\mathbf{X} \rightarrow \mathbf{E}\} \propto \prod_{i=1}^{r} \left( 1 + \frac{E_s}{4N_0} \lambda_i \right)^{-N_R} \left( \prod_{i=1}^{r} \lambda_i \right)^{1/r} \cdot \left( \frac{E_s}{4N_0} \right)^{-r \cdot N_R}
\]

- **Diversity gain** determines the slope of BER curve in log-scale

\[g_D = r \cdot N_R\]

- **Coding gain** determines horizontal shift

\[g_C = \min_{(\mathbf{X}, \mathbf{E})} \left( \prod_{i=1}^{r} \lambda_i \right)^{1/r}\]

For difference matrix of full rank \((r = N_T)\)

\[g_C = \min_{(\mathbf{X}, \mathbf{E})} (\det \Delta(\mathbf{X}, \mathbf{E}))^{1/N_T}\]
Design Criteria for Space-Time Codes

- **Rank Criterion:**
  In order to achieve the maximum diversity $N_T \cdot N_R$, the difference matrix $(X-E)$ has to be full rank for any codeword matrices $X$ and $E$. If $(X-E)$ has a minimum rank $r$ over the set of pairs of distinct words, a diversity of $r \cdot N_R$ is achieved.

\[
\min \text{rank} \{ \Delta(X, E) \} = N_T \quad \iff \quad g_D = \min \text{rank} \{ \Delta(X, E) \} \cdot N_R = N_T \cdot N_R
\]

- **Determinant Criterion:**
  In order to achieve the maximum coding gain for a given diversity gain of $N_T \cdot N_R$, maximize the minimum product of eigenvalues for any two codeword matrices $X$ and $E$.

\[
g_C = \min_{(X,E)} \left( \prod_{i=1}^{r=N_T} \right)^{1/N_T} = \min_{(X,E)} (\det \Delta(X, E))^{1/N_T}
\]
Orthogonal Space-Time Blockcodes (OSTBC)
Orthogonal Space-Time Blockcodes

- Alamouti’s scheme
  - Transmission scheme for $N_T = 2$ antennas
  - Equivalent to MRC with 2 antennas at receiver

- Generalization by Tarokh for more than 2 transmit antennas
  - Orthogonal Space-Time Blockcodes

- Simple modulation scheme for limited number of transmit antennas
- Easy detection (demodulation) by linear combination of the received signals
- Transmit diversity schemes can be combined with multiple receive antennas!
Alamouti's Scheme (1)

- Code word matrix of two consecutive time steps
  \[
  X = \begin{bmatrix}
  x_1[k] & x_1[k+1] \\
  x_2[k] & x_2[k+1]
  \end{bmatrix} = \begin{bmatrix}
  s_1 & -s_2^* \\
  s_2 & s_1^*
  \end{bmatrix}
  \]

- Received signal vector of one block
  \[
  Y = HX + N
  \]
  \[
  \begin{bmatrix}
  y_1[k] & y_1[k+1]
  \end{bmatrix} = \begin{bmatrix}
  h_{1,1} & h_{1,2}
  \end{bmatrix} \cdot \begin{bmatrix}
  s_1 & -s_2^* \\
  s_2 & s_1^*
  \end{bmatrix} + \begin{bmatrix}
  n_1[k] & n_1[k+1]
  \end{bmatrix}
  \]
  \[
  = \begin{bmatrix}
  h_{1,1}s_1 + h_{1,2}s_2 & -h_{1,1}s_2^* + h_{1,2}s_1^*
  \end{bmatrix} + \begin{bmatrix}
  n_1[k] & n_1[k+1]
  \end{bmatrix}
  \]
  \[
  \begin{bmatrix}
  y_1[k] & y_1^*[k+1]
  \end{bmatrix} = \begin{bmatrix}
  h_{1,1}s_1 + h_{1,2}s_2 \\
  -h_{1,1}^*s_2 + h_{1,2}^*s_1
  \end{bmatrix} + \begin{bmatrix}
  n_1[k] & n_1^*[k+1]
  \end{bmatrix} = \begin{bmatrix}
  h_{1,1} & h_{1,2} \\
  h_{1,2}^* & -h_{1,1}^*
  \end{bmatrix} \begin{bmatrix}
  s_1 \\
  s_2
  \end{bmatrix} + \begin{bmatrix}
  n_1[k] \\
  n_1^*[k+1]
  \end{bmatrix}
  \]
Alamouti’s Scheme (2)

- Linear combining is matched filtering

\[
\begin{bmatrix}
\tilde{s}_1 \\
\tilde{s}_2
\end{bmatrix} =
\begin{bmatrix}
h_{1,1} & h_{1,2} \\
h_{1,2} & -h_{1,1}
\end{bmatrix}^H
\begin{bmatrix}
y_1[k] \\
y_1^*[k + 1]
\end{bmatrix} =
\begin{bmatrix}
h_{1,1} & h_{1,2} \\
h_{1,2} & -h_{1,1}
\end{bmatrix}^H
\begin{bmatrix}
h_{1,1} & h_{1,2} \\
h_{1,2} & -h_{1,1}
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2
\end{bmatrix} +
\begin{bmatrix}
h_{1,1} & h_{1,2} \\
h_{1,2} & -h_{1,1}
\end{bmatrix}^H
\begin{bmatrix}
n_1[k] \\
n_1[k + 1]
\end{bmatrix} =
\begin{bmatrix}
h_{1,1} & h_{1,2} \\
h_{1,2} & -h_{1,1}
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2
\end{bmatrix} +
\begin{bmatrix}
n_1[k] \\
n_1[k + 1]
\end{bmatrix}
\]

\[
(|h_{1,1}|^2 + |h_{1,2}|^2) \cdot I_2
\]

- Modified received signal vector after linear combining

\[
\begin{bmatrix}
\tilde{s}_1 \\
\tilde{s}_2
\end{bmatrix} = (|h_{1,1}|^2 + |h_{1,2}|^2) \cdot \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{bmatrix} = \tilde{s} = ||H||^2 \cdot s + \tilde{n}
\]

Diversity degree \( g_D = 2 \! \)

- Two independent signals \( \tilde{s}_1 \) and \( \tilde{s}_2 \) to represent \( s_1 \) and \( s_2 \)
- Code rate: \( R_c^{ST} = 1 \) (2 symbols in 2 time slots)
- Independent detection of signals on basis of linear combining allows very simple receiver structure!
General Remarks on Orthogonal STBC (1)

- General Results from matrix theory
  - Orthogonal matrices with complex elements for code rate 1 exists only for $N_T = 2$ antennas → Alamouti
  - Orthogonal matrices with complex elements for code rate 1/2 exist for any number of transmit antennas
  - Orthogonal matrices with complex elements for code rate 3/4 exist for $N_T = 3$ and $N_T = 4$ antennas
  - Orthogonal quadratic matrices with real valued elements for code rate 1 exist only for $N_T = 2$, $N_T = 4$ and $N_T = 8$

- System with $N_T$ transmit antennas
  - Transmission of $m$ different information symbols $s_m$
  - Occupation of $p$ time slots for transmission
  - Description by $N_T \times p$ code matrix $G_{NT}$

Part 4: Space-Time Signal Processing
General Remarks on Orthogonal STBC (2)

- Space-Time code rate: $m$ symbols are transmitted in $p$ timeslots
- Spectral efficiency $R_c^{ST} \cdot \log_2(M) \text{Bit/s/Hz}$

- Elements of $G_{NT}$ are given by **linear combinations** of the variables $0, s_1, s_1^*, s_2, s_2^*, \cdots, s_m, s_m^*$ → **STBC are linear codes**
  - Only with conjugated elements linear description is possible because conjugation is no linear transformation
  - Code matrix $G_{NT}$ consists of orthogonal rows

$$G_{NT} \cdot G_{NT}^H = (|s_1|^2 + |s_2|^2 + \cdots + |s_m|^2) \cdot I_{NT}$$

- Alternatively, real valued description with twice as large matrices possible

$$G_{NT} (0, s_1, s_1^*, s_2, s_2^*, \cdots, s_m, s_m^*) \rightarrow \tilde{G}_{NT} (0, s'_1, s''_1, s'_2, s''_2, \cdots, s'_m, s''_m)$$
Real and Complex Representation of Alamouti’s Scheme

- Alamouti’s scheme with $N_T = 2$, $m = 2$, $p = 2$: $R_c^{ST} = \frac{m}{p} = 1$

$$G_2 = G_2(0, s_1, s_1^*, s_2, s_2^*) = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}$$

$$y = \begin{bmatrix} y[k] \\ y[k+1] \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} n[k] \\ n[k+1] \end{bmatrix}$$

- Real representation

$$\tilde{G}_2 = \tilde{G}_2(s_1', s_1'', s_2', s_2'') = \begin{bmatrix} s_1' & s_2' & -s_1'' & -s_2'' \\ -s_2' & s_1' & s_2'' & -s_1'' \\ s_1'' & s_2'' & s_1' & s_2' \\ -s_2'' & s_1'' & -s_2' & s_1' \end{bmatrix} = \begin{bmatrix} \text{Re} \\ -\text{Im} \\ \text{Im} \\ \text{Re} \end{bmatrix}$$

$$y' = \begin{bmatrix} y'[k] \\ y'[k+1] \\ y''[k] \\ y''[k+1] \end{bmatrix} = \begin{bmatrix} s_1' & s_2' & -s_1'' & -s_2'' \\ -s_2' & s_1' & s_2'' & -s_1'' \\ s_1'' & s_2'' & s_1' & s_2' \\ -s_2'' & s_1'' & -s_2' & s_1' \end{bmatrix} \cdot \begin{bmatrix} h'_1 \\ h'_2 \\ h''_1 \\ h''_2 \end{bmatrix} + \begin{bmatrix} n'[k] \\ n'[k+1] \\ n''[k] \\ n''[k+1] \end{bmatrix}$$
Orthogonal STBC for Rate 1/2, $N_T=3$

- $N_T=3$ antennas, $m = 4$ information symbols, $p = 8$ time slots

\[ G_3 = \begin{bmatrix} s_1 & -s_2 & -s_3 & -s_4 & s_1^* & -s_2^* & -s_3^* & -s_4^* \\ s_2 & s_1 & s_4 & -s_3 & s_2^* & s_1^* & s_4^* & -s_3^* \\ s_3 & -s_4 & s_1 & s_2 & s_3^* & -s_4^* & s_1^* & s_2^* \end{bmatrix} \]

\[ R_{ST}^c = \frac{m}{p} = \frac{4}{8} = 0.5 \]

- Rows of code matrix $G_3$ are orthogonal

\[ G_3 \cdot G_3^H = (|s_1|^2 + |s_2|^2 + |s_3|^2 + |s_4|^2) \cdot I_3 \]

- Receive vector $y$ of dimension 1x8

\[ y = H \cdot G_3 + n = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \end{bmatrix} \cdot \begin{bmatrix} s_1 & -s_2 & -s_3 & -s_4 & s_1^* & -s_2^* & -s_3^* & -s_4^* \\ s_2 & s_1 & s_4 & -s_3 & s_2^* & s_1^* & s_4^* & -s_3^* \\ s_3 & -s_4 & s_1 & s_2 & s_3^* & -s_4^* & s_1^* & s_2^* \end{bmatrix} + n \]

\[ y = \begin{bmatrix} h_{1,1}s_1 + h_{1,2}s_2 + h_{1,3}s_3 \\ \cdots [-h_{1,1}s_4 - h_{1,2}s_3 + h_{1,3}s_2] \\ \cdots [-h_{1,1}s_3^* + h_{1,2}s_4^* + h_{1,3}s_1^*] \end{bmatrix} + \begin{bmatrix} -h_{1,1}s_2 + h_{1,2}s_1 - h_{1,3}s_4 \\ h_{1,1}s_1^* + h_{1,2}s_2^* + h_{1,3}s_3^* \\ -h_{1,1}s_4^* - h_{1,2}s_3^* + h_{1,3}s_2^* \end{bmatrix} + n \]
Orthogonal STBC for Rate 1/2, $N_T = 3$

- Generate modified receive vector by conjugation of signals received in time instances $5,\ldots, 8$

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5^* \\
y_6^* \\
y_7^* \\
y_8^*
\end{bmatrix}
= 
\begin{bmatrix}
h_1 & h_2 & h_3 & 0 \\
h_2 & -h_1 & 0 & -h_3 \\
h_3 & 0 & -h_1 & h_2 \\
0 & h_3 & -h_2 & -h_1 \\
h_1^* & h_2^* & h_3^* & 0 \\
h_2^* & -h_1^* & 0 & -h_3^* \\
h_3^* & 0 & -h_1^* & h_2^* \\
0 & h_3^* & -h_2^* & -h_1^*
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4
\end{bmatrix}
= \begin{bmatrix}
n_1 \\
n_2 \\
n_3 \\
n_4
\end{bmatrix}
\]

- Equivalent channel matrix $\tilde{H}$ contains orthogonal columns
- Modified received signal vector after linear combining

\[
\tilde{s} = \tilde{H}^H \cdot \tilde{y} = \tilde{H}^H \cdot \tilde{H} \cdot s + \tilde{H}^H \tilde{n} = ||\tilde{H}||^2 s + \tilde{n}
\]
Orthogonal STBC for Rate 1/2, \( N_T = 4 \)

- \( N_T = 4 \) antennas, \( m = 4 \) information symbols, \( p = 8 \) time slots

\[
G_4 = \begin{bmatrix}
  s_1 & -s_2 & -s_3 & -s_4 & s_1^* & -s_2^* & -s_3^* & -s_4^* \\
  s_2 & s_1 & s_4 & -s_3 & s_2^* & s_1^* & s_4^* & -s_3^* \\
  s_3 & -s_4 & s_1 & s_2 & s_3^* & -s_4^* & s_1^* & s_2^* \\
  s_4 & s_3 & -s_2 & s_1 & s_4^* & s_3^* & -s_2^* & s_1^*
\end{bmatrix}
\]

- Modified receive vector \( \rightarrow \) equivalent channel matrix \( \tilde{H} \) with orthogonal columns

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5^* \\
y_6^* \\
y_7^* \\
y_8^*
\end{bmatrix} = \begin{bmatrix}
h_1 & h_2 & h_3 & h_4 \\
h_2 & -h_1 & h_4 & -h_3 \\
h_3 & -h_4 & -h_1 & h_2 \\
h_4 & h_3 & -h_2 & -h_1 \\
h_1^* & h_2^* & h_3^* & h_4^* \\
h_2^* & -h_1^* & h_4^* & -h_3^* \\
h_3^* & -h_4^* & -h_1^* & h_2^* \\
h_4^* & h_3^* & -h_2^* & -h_1^*
\end{bmatrix} \cdot \begin{bmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4
\end{bmatrix} = \begin{bmatrix}
n_1 \\
n_2 \\
n_3 \\
n_4 \\
n_5^* \\
n_6^* \\
n_7^* \\
n_8^*
\end{bmatrix}
\]

\[
R_{c}^{ST} = \frac{m}{p} = \frac{4}{8} = 0.5
\]
Orthogonal STBC for Rate 3/4

- $N_T = 3$ antennas, $m = 3$ information symbols, $p = 4$ time slots

$$H_3 = \begin{bmatrix}
s_1 & -s_2^* & \frac{1}{\sqrt{2}}s_3^* & \frac{1}{\sqrt{2}}s_3 \\
- s_2 & s_1^* & \frac{1}{\sqrt{2}}s_3^* & -\frac{1}{\sqrt{2}}s_3 \\
\frac{1}{\sqrt{2}}s_3 & \frac{1}{\sqrt{2}}s_3 & \frac{1}{2}(-s_1 - s_1^* + s_2 - s_2^*) & \frac{1}{2}(s_1 - s_1^* + s_2 + s_2^*)
\end{bmatrix}$$

- $N_T = 4$ antennas, $m = 3$ information symbols, $p = 4$ time slots

$$H_3 = \begin{bmatrix}
s_1 & -s_2^* & \frac{1}{\sqrt{2}}s_3^* & \frac{1}{\sqrt{2}}s_3 \\
- s_2 & s_1^* & \frac{1}{\sqrt{2}}s_3^* & -\frac{1}{\sqrt{2}}s_3 \\
\frac{1}{\sqrt{2}}s_3 & \frac{1}{\sqrt{2}}s_3 & \frac{1}{2}(-s_1 - s_1^* + s_2 - s_2^*) & \frac{1}{2}(s_1 - s_1^* + s_2 + s_2^*) \\
\frac{1}{\sqrt{2}}s_3 & -\frac{1}{\sqrt{2}}s_3 & \frac{1}{2}(s_1 - s_1^* - s_2 - s_2^*) & -\frac{1}{2}(s_1 + s_1^* + s_2 - s_2^*)
\end{bmatrix}$$
Simulation Results for STBC (1)

- **Simulation parameters**
  - Alamouti-STBC, $N_T = 2$,
  - $G_2 = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}$
  - 2 symbols in 2 time slots $\rightarrow R_c^{ST} = 1$
  - QPSK $\rightarrow$ 2 bits / time slot

- **Results**
  - **Diversity gain** determines slope of BER
  - Diversity of $N_T \cdot N_R$ results in strong performance improvement
Simulation Results for STBC (2)

- Simulation parameters
  - STBC for $N_T = 3$,
  - $3$ symbols in $4$ time slots $\rightarrow R_c = 3/4$
  - QPSK $\rightarrow 1.5$ bits / slot

- Result
  - Increased diversity in comparison to Alamouti due to $N_T \cdot N_R$

$H_3 = \begin{bmatrix}
s_1 & -s_2^* & 1/\sqrt{2}s_3^* & 1/\sqrt{2}s_3^* \\
s_2 & s_1^* & 1/\sqrt{2}s_3^* & -1/\sqrt{2}s_3^* \\
1/\sqrt{2}s_3 & 1/\sqrt{2}s_3 & -s_1'' + s_2'' & s_1'' + s_2''
\end{bmatrix}$
Selected References for STBC

- **Paper**

- **Online books:**

- **Printed books:**
  - V. Kühn: Wireless Communications over MIMO Channels, Wiley, 2006
  - M. Jankiraman: Space-Time Codes and MIMO Systems, Artech House, 2004
Space-Time Trellis Codes (STTC)

- Orthogonal Space-Time Block Codes do not achieve any coding gain
- Exploit diversity as well as coding gain by applying trellis codes
- Non-orthogonal block codes also possible, but not subject of this course
Space-Time Trellis-Codes: Delay Diversity

- Properties of delay diversity
  - Transmit delayed replicas of the same signal from different antennas
  - Flat MISO channel transformed into frequency selective SISO channel
  - Equalization of received signal, e.g. by Viterbi equalizer
  - Maximum diversity $N_T \cdot N_R$ is achieved
  - Drawback: computational costs grow exponentially with diversity order

- Are there better codes than repetition code?

\[
\begin{align*}
  x_1[k] &= s[k] \\
  x_2[k] &= s[k - 1] \\
  x_{NT}[k] &= s[k - NT + 1] \\
  y[k] &= \sum_{i=1}^{NT} h_i s[k - i + 1] + n[k] \\
  \hat{s}[k] &= \text{equalizer}\end{align*}
\]
Encoder Structure for Delay Diversity

- Nonrecursive convolutional encoder realized by binary shift registers
- Example: Delay-Diversity for QPSK, $N_T = 2$ transmit antennas, 4 states
  - Binary register elements ($b_i[k] \in \{0,1\}$)
    \[
    a[k] = \begin{bmatrix} b[k] \\ b[k-1] \end{bmatrix} = \begin{bmatrix} b_1[k] \\ b_2[k] \\ b_1[k-1] \\ b_2[k-1] \end{bmatrix}
    \]
  - Generator matrix
    \[
    G = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}
    \]
  - Encoder output: ($c_i[k] \in \{0,1,2,3\}$)
    \[
    c[k] = \begin{bmatrix} c_1[k] \\ c_2[k] \end{bmatrix} = (G \cdot a[k]) \mod 4
    \]
  - Transmit vector (QPSK symbols with natural mapping):
    \[
    x[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} = M\{c[k]\}
    \]
Trellis Representation for Delay Diversity

- **Trellis structure for delay diversity**
  
  - Output $c[k]$:
    - $00, 10, 20, 30$
    - $01, 10, 21, 31$
    - $02, 12, 22, 32$
    - $03, 13, 23, 33$

  - Input $b[k]$:
    - $00$
    - $10$
    - $01$
    - $11$

  - State $b[k - 1]$
    - $00 \rightarrow \zeta_0$
    - $10 \rightarrow \zeta_1$
    - $01 \rightarrow \zeta_2$
    - $11 \rightarrow \zeta_3$

  - **QPSK constellation**

- **Example**

  - Input $b = [01 \ 11 \ 10 \ 00]^T$

  - **Output**
    
    $C = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$

  - **Code word matrix**
    
    $X = \begin{bmatrix} -1 & -j & j & 1 \\ 1 & -1 & -j & j \end{bmatrix}$

---

Part 4: Space-Time Signal Processing
General Encoder Structure for STTC

- Example for 8-PSK, $N_T = 2$ and shift register with memory $\ell$

Generator matrix

$$G = \begin{bmatrix} G_{1,1} & G_{1,2} & \ldots & G_{1,4\ell} \\ G_{2,1} & G_{2,2} & \ldots & G_{2,4\ell} \end{bmatrix}$$

Encoder output:

$$c[k] = \begin{bmatrix} c_1[k] \\ c_2[k] \end{bmatrix} = (G \cdot a[k]) \mod 8$$

Transmit vector:

$$x[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} = M\{c[k]\}$$
Code Search for Space-Time Trellis Codes

- Different codes for different configurations
  - Number of transmit antennas
  - Order of modulation
  - Length of register → number of states

- Codes for same configuration differ only in generator coefficients

- Systematic code search by calculating diversity gain and coding gain for all permutations of $G$
  - Look only for Space-Time Trellis Codes with maximum diversity
  - Choose the code with highest coding gain among those with maximum diversity
  - Best (known) Space-Time Trellis Code for 2 transmit antennas, QPSK, 4 states found by Yan and Blum, Lehigh University

$$G_{opt} = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}$$
Simulation Results for STTC

- **Simulation parameters**
  - $N_T = 2$, 4-PSK, 100 symbols

- **Results**
  - **Diversity gain** determines slope of FER
  - **Coding gain** affects horizontal shift for codes of same diversity
  - Performance of STTC of same constellation differ only for $N_R > 1$
  - Increased coding gain with larger number of states, but also higher decoding effort
Cyclic Delay Diversity for OFDM

- Transmission of cyclic shifted version of same OFDM symbol
- Frequency selectivity of channel increased (can only be exploited by channel coding)
- Approach consistent with standard: no modification of receiver required

channel transfer function

\[ |C|_{\text{dB}} \]

\[ \text{subcarrier } n \]

channel transfer function

\[ |C|_{\text{dB}} \]

\[ \text{subcarrier } n \]
Selected References for STTC

Paper

- S. Bäro: Performance Analysis of Space-Time Trellis Coded Modulation of Flat Fading Channels, ITG-Diskussionssitzung Systeme mit intelligenten Antennen, Stuttgart, Germany, April 1999

Online books:


Books

- V. Kühn: Wireless Communications over MIMO Channels, Wiley, 2006
- M. Jankiraman: Space-Time Codes and MIMO Systems, Artech House, 2004

Part 4: Space-Time Signal Processing
Layered Space-Time Codes (BLAST)

Transmission of multiple (parallel) data streams for higher data rates without increasing bandwidth
V-BLAST Transmitter

- **V-BLAST → V**ertical B**ell-Labs L**Ayered S**pace-T**ime Architecture
  - Transmitted code words (layers) are vertically arranged
  - Each layer is transmitted over one particular antenna

**Structure of transmit signal (example: four transmit antennas)**

- either uncoded or individually encoded, multiple users possible

Part 4: Space-Time Signal Processing
D-BLAST Transmitter

- **D-BLAST → Diagonal Bell-Labs LAyered Space-Time Architecture**
  - Transmitted code words (layers) are distributed over all antennas
  - Higher diversity gain for each layer compared to V-BLAST (with FEC coding)

- **Structure of transmit signal (example: four transmit antennas)**
  - Each layer is individually encoded, multiple users possible
Receiver for the V-BLAST Scheme

- Optimal Detection Scheme
  - Maximum-Likelihood Detection

- Linear Equalizer
  - Zero-Forcing Criterion
  - Minimum Mean Square Error Criterion

- Successive Interference Cancellation
  - V-BLAST Detection Algorithm
  - SIC on bases of Sorted QR Decomposition
  - Post Sorting Algorithm

- Sphere Detection
Optimal Detection

- Received signals are superposition of all transmit signals (plus noise)

\[ y = \sum_{i=1}^{N_T} h_i x_i + n = H \cdot x + n \]

- Optimum detector fulfills maximum likelihood (ML) criterion
  - Coded transmission:
    Find set of code words (sequences → MLSE) that was transmitted most likely
    → Extremely high computational complexity
  - Uncoded transmission:
    Find set of symbols that was transmitted most likely
    → Solve linear equation system with respect to the discrete symbol alphabet
    → Still very high computational complexity
Optimal Detection

- **Maximum-Likelihood (ML)**
  
  \[
  \hat{x}_{\text{ML}} = \arg \max_{x \in A^{NT}} p(y|H, x) = \arg \min_{x \in A^{NT}} \|y - Hx\|^2
  \]

- Brute Force:
  Find *minimum* Euclidian distance over all \( x \in A^{NT} \)
  
  \( \rightarrow \) Effort grows exponentially with spectral efficiency

  Example: \( N_T=4 \) and 16-QAM: \( M^{N_T} = 2^{\log_2(M)N_T} = 16^4 = 65536 \)

- More efficient implementation by Sphere-Detection (SD):
  - Efficient algorithm with low best case complexity
  - Still high worst case complexity

- Less complex detection:
  - Linear processing
  - Suboptimal non-linear processing
Linear Equalizer (Linear Detector, LD)

- Derivation of the filter matrix $G$
  - Error vector
    \[ e = \hat{x} - x = GHn + Gn - x = (GH - I_{NT})x + Gn \]
  - Error covariance matrix diagonal elements determine layer-specific errors
    \[ \Phi_{ee} = E\{ee^H\} = E\{ (\hat{x} - x)(\hat{x} - x)^H \} = (GH - I_{NT})(GH - I_{NT})^H + G\Phi_{nn}G^H \]
  - Average power of the estimation error is given by
    \[ E\{\|e\|\} = \text{tr}\{\Phi_{ee}\} \]
  - General form of the filter output signal
    \[ \hat{x} = \text{dg}\{GH\} \cdot x + \text{dg}\{GH\} \cdot x + Gn \]
Linear Equalizer (Linear Detector, LD)

- Linear filtering of receive signals
- Quantization per layer

- Derivation of the filter matrix \( G \)
  - General form of the filter output signal
  \[
  \tilde{x} = \text{dg}\{GH\} \cdot x + \text{dg}\{GH\} \cdot x + Gn
  \]

- Signal-to-Interference-and-Noise-Ratio (SINR)
  \[
  \text{SINR}_i = \frac{P_{S,i}}{P_{I,i} + P_{N,i}} = \frac{P_{S,i}}{P_{T,i} - P_{S,i}} = \frac{P_{S,i}}{1 - \frac{P_{S,i}}{P_{T,i}}}
  \]

- \( P_{S,i} = \mathbb{E}\{|\text{dg}\{GH\} \cdot x\}_i|^2\} \)
- \( P_{I,i} = \mathbb{E}\{|\text{dg}\{GH\} \cdot x\}_i|^2\} \)
- \( P_{N,i} = \mathbb{E}\{|G \cdot n\}_i|^2\)
Linear Equalizer by Inversion

- Inversion of receive signal
- Estimation of transmitted symbol

\[ \tilde{x} = H^{-1} \cdot y \]

\[ \hat{x}_i = Q\{\tilde{x}_i\} \]

**Signal space diagrams**
- **Receive signal** $y$
- **Filter output signal** $\tilde{x}$

Unreliable estimation
Linear Zero-Forcing Equalizer

- Zero-Forcing Criterion \( \rightarrow \) “Least Square Solution”
  - Minimize the Euclidian distance \( ||y - H\tilde{x}||^2 \)
  - \( G \) is given by Pseudo-Inverse of channel matrix
    \[
    G_{ZF} = H^+ = (H^H H)^{-1} H^H
    \]
  - Projection of the received signal \( y \) onto the \( N_T \)-dim. subspace spanned by \( H \) within the \( N_R \)-dimensional receive space
  - Perfectly suppresses mutual interference \( \rightarrow \) Problem: Noise enhancement!
    \[
    \tilde{x} = G_{ZF} \cdot y = H^+ H x + H^+ n = x + \tilde{n}
    \]
  - Error covariance matrix
    \[
    \Phi_{ee, ZF} = E\{\tilde{n}\tilde{n}^H\} = \sigma_n^2 G_{ZF} G_{ZF}^H = \sigma_n^2 H^+ H^+ = \sigma_n^2 (H^H H)^{-1}
    \]
- SNR
  \[
  SNR_{ZF, i} = \frac{P_{S, i}}{P_{N, i}} = \frac{1}{\Phi_{ee, ZF}_{i,i}} = \frac{1}{\sigma_n^2 \|g_{ZF}^{(i)}\|^2}
  \]
Linear MMSE Equalizer

- **Minimum-Mean-Square-Error Criterion (MMSE)**
  - Minimization of the mean error at the filter output
  - By introducing the receive covariance matrix $\Phi_{yy} = \mathbb{E}\{yy^H\} = HH^H + \Phi_{nn}$
    - the error covariance matrix can be rewritten in quadratic form

\[
\Phi_{ee} = \mathbb{E}\{(Gy - x)(Gy - x)^H\} = \mathbb{E}\{Gyy^H G^H - Gyx^H - xy^H G^H + xx)^H\}
\]

\[
= G\Phi_{yy} G^H - GH - H^H G^H + I_{N_T}
\]

\[
= (G\Phi_{yy} - H^H) \Phi_{yy}^{-1} (G\Phi_{yy} - H^H)^H - H^H \Phi_{yy}^{-1} H + I_{N_T}
\]

- $\Phi_{yy}$ is non-negative definite $\rightarrow$ trace of first term can not be negative
- Minimum for $G\Phi_{yy} - H^H = 0_{N_T, N_T}$
  - $\Phi_{nn} = \sigma^2_n I_{N_R}$
- Solution for the filter matrix

\[
G = H^H \cdot \Phi_{yy}^{-1} = H^H (HH^H + \sigma^2_n I_{N_R})^{-1} = (H^H H + \sigma^2_n I_{N_T})^{-1} H^H
\]

- Error covariance matrix

\[
\Phi_{ee, MMSE} = I_{N_T} - H^H \Phi_{yy}^{-1} H = \sigma^2_n (H^H H + \sigma^2_n I_{N_T})^{-1}
\]
Linear MMSE Equalizer (2)

- MMSE Filter output signal (biased estimator)
  \[ \tilde{x}_{\text{MMSE}} = H^H \Phi_{yy}^{-1} y = H^H (HH^H + \sigma_n^2 I_{NR})^{-1} y \]

- \(i\)-th filter output signal
  \[ \tilde{x}_i = h_i^H \Phi_{yy}^{-1} y = h_i^H \Phi_{yy}^{-1} (Hx + n) = h_i^H \Phi_{yy}^{-1} h_i x_i + h_i^H \Phi_{yy}^{-1} h_i x_i + h_i^H \Phi_{yy}^{-1} n \]

- SINR
  \[ \text{SINR}_{\text{MMSE},i} = \frac{P_{S,i}}{P_{I,i} + P_{N,i}} = \frac{h_i^H \Phi_{yy}^{-1} h_i}{1 - h_i^H \Phi_{yy}^{-1} h_i} = \frac{1 - [\Phi_{\text{ee},\text{MMSE}}]_{i,i}}{[\Phi_{\text{ee},\text{MMSE}}]_{i,i}} = \frac{1}{[\Phi_{\text{ee},\text{MMSE}}]_{i,i}} - 1 \]

- Unbiased estimator
  - Assume channel matrix with orthogonal columns
  - Bias leads to amplitude scaling \(\rightarrow\) important for QAM
  - Solutions:
    - Adopt filter \( G_{\text{UB-MMSE}} = (dg\{G_{\text{MMSE}}H\})^{-1} G_{\text{MMSE}} \)
    - Consider scaling within the demodulator
Linear MMSE Equalizer (3)

- Relation of MMSE to zero-forcing
  - Definition of extended channel matrix and extended receive vector
    \[ \mathbf{H} = \begin{bmatrix} \mathbf{H} \\ \sigma_n \mathbf{I}_{NT} \end{bmatrix} \Rightarrow \mathbf{H}^H \mathbf{H} = \mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}_{NT} \]
  - Applying zero-forcing approach to \( \mathbf{H} \) leads to MMSE solution with \( \mathbf{H} \)
    - Filter output expressed with \( \mathbf{H} \) and \( \mathbf{y} \)
      \[ \tilde{\mathbf{x}}_{\text{MMSE}} = (\mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}_{NT}) \mathbf{H}^H \mathbf{y} = \left( \begin{bmatrix} \mathbf{H}^H & \sigma_n \mathbf{I}_{NT} \end{bmatrix} \begin{bmatrix} \mathbf{H} \\ \sigma_n \mathbf{I}_{NT} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{H}^H & \sigma_n \mathbf{I}_{NT} \end{bmatrix} \mathbf{y} \]
      \[ = (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{y} = \mathbf{H}^+ \mathbf{y} \]
    - Error covariance matrix expressed with \( \mathbf{H} \)
      \[ \Phi_{\text{MMSE}} = \sigma_n^2 (\mathbf{H}^H \mathbf{H})^{-1} = \sigma_n^2 \cdot \mathbf{H}^+ \mathbf{H}^+ \]
    - MMSE solution corresponds to zero-forcing for extended system

Part 4: Space-Time Signal Processing
Bit Error Rate for Linear Equalization

- **Simulation parameters**
  - $N_T = 4$, 4-QAM

- **Results**
  - Linear Equalization leads to strong performance drawback in comparison to ML
  - MMSE outperforms ZF
  - With increased $N_R$ the slope of BER increases (receive diversity) and gap between linear and ML becomes smaller
Successive Interference Cancellation

- Basic principle of Successive Interference Cancellation (SIC)
  - Cancel estimated interference of already detected layer and linearly suppress the interference of remaining layer
  - Optimization of detection sequence to reduce error propagation

- V-BLAST Algorithm
  - Based on linear ZF or MMSE equalization
  - In each step only the layer with maximum SNR / SINR is detected

\[ y \rightarrow \text{Detection 1. Layer} \rightarrow \hat{x}_{k_1} \]

\[ \hat{x}_{k_1} \times h_{k_1} \rightarrow \text{Detection 2. Layer} \rightarrow \hat{x}_{k_2} \]

\[ \hat{x}_{k_2} \times h_{k_2} \rightarrow \text{Reduce H} \]

\[ k_i \text{ indicates layer estimated in detected step } i \]
V-BLAST Detection Algorithm (2)

- **General procedure**
  - Apply zero forcing only for one layer (nulling interfering users)
  - Detect best layer and subtract estimated interference
  - Continue with next layer until all layers have been processed
- **Order of detection is crucial → sorting criterion is necessary**
  - Error covariance matrix: diagonal elements determine layer-specific errors

\[
\Phi_{ZF} = E\{(\tilde{x}_{ZF} - x)(\tilde{x}_{ZF} - x)^H\}
= E\{(x + G_{ZF}n - x)(x + G_{ZF}n - x)^H\}
= \sigma_n^2 \cdot G_{ZF} \cdot G_{ZF}^H = \sigma_n^2 \cdot (H^H H)^{-1}
\]

- Layer corresponding to smallest diagonal element in \(\Phi_{ZF}\) has smallest error
  - Row \(g_{ZF}^{(i)}\) of \(G_{ZF}\) with smallest squared norm corresponds to minimum diagonal element in \(\Phi_{ZF}\)
  - Smallest noise amplification → best SNR
V-BLAST Detection Algorithm (3)

- Detailed procedure:
  - Determine layer with smallest noise amplification (best SNR)
  - Apply linear filtering to layer $k_i$
    \[
    \tilde{x}_{k_i} = g_{ZF}^{(k_i)} \cdot y = x_{k_i} + g_{ZF}^{(k_i)} n = x_{k_i} + \left[(H^H H)^{-1} H^H\right]_{k_i} n
    \]
  - Detect layer after filtering, i.e. find estimate $\hat{x}_{k_i}$ for $x_{k_i}$ by quantization of $\tilde{x}_{k_i}$
  - Subtract estimated interference from receive signal
    \[
    y \leftarrow y - h_{k_i} \cdot \hat{x}_{k_i}
    \]
  - Remove $i$-th column from channel matrix
    \[
    H \leftarrow \begin{bmatrix}
    h_1 & \cdots & h_{k_i-1} & h_{k_i+1} & \cdots & h_{N_T}
    \end{bmatrix}
    \]
  - Continue with next layer of reduced system until all layers have been detected

Expensive calculation of pseudo-inverse in each iteration
SIC with QR Decomposition (1)

- Costly calculation of pseudo inverse should be avoided
- Applying QR decomposition of $H$
  \[ H = QR \]
  - $Q$ is a $N_R \times N_T$ matrix with orthogonal columns of unit length
  - $R$ is a $N_T \times N_T$ upper triangular matrix
- Multiplication of $y$ with $Q^H$ delivers starting point for successive interference cancellation without any further linear filtering
  \[
  \tilde{x} = Q^H y = Q^H Q R x + Q^H n = R x + \eta
  \]
- Only 1 QR decomposition instead of calculating $N_T - 1$ pseudo inverses
- However, sorting is still an open problem
**SIC with QR Decomposition (2)**

- Linear filtering of $y$ with $Q^H$ yields
  \[
  \begin{pmatrix}
  \tilde{x}_1 \\
  \vdots \\
  \tilde{x}_k \\
  \tilde{x}_{k+1} \\
  \vdots \\
  \tilde{x}_{N_T}
  \end{pmatrix}
  =
  \begin{pmatrix}
  r_{1,1} & \cdots & r_{1,k} & r_{1,k+1} & \cdots & r_{1,N_T} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & r_{k,k} & r_{k,k+1} & \cdots & r_{k,N_T} \\
  0 & \cdots & 0 & r_{k+1,k+1} & \cdots & r_{k+1,N_T} \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & 0 & \cdots & r_{N_T,N_T}
  \end{pmatrix}
  \begin{pmatrix}
  x_1 \\
  \vdots \\
  x_k \\
  x_{k+1} \\
  \vdots \\
  x_{N_T}
  \end{pmatrix}
  +
  \begin{pmatrix}
  \eta_1 \\
  \vdots \\
  \eta_k \\
  \eta_{k+1} \\
  \vdots \\
  \eta_{N_T}
  \end{pmatrix}
  \]

- Layer $k$ experiences only interference from layers $k + 1, \ldots, N_T$

  \[
  \tilde{x} = r_{k,k} \cdot x_k + \sum_{i=k+1}^{N_T} r_{k,i} \cdot x_i + \eta_k
  \]

  - Signal $\tilde{x}_{N_T}$ is free of interference and can be directly decided
  - Subtract estimated interference from other layers and continue detection with $\tilde{x}_{N_T-1}$ until first layer $\tilde{x}_1$ has been decided
  - SNR in layer $N_T$: $\text{SNR}_{N_T} = \sigma_n^{-2} |r_{N_T,N_T}|^2$

---

**Part 4: Space-Time Signal Processing**
SIC with QR Decomposition (3)

- **Interference Cancellation**
  \[
  \tilde{x}_k^{IC} = \tilde{x}_k - \sum_{i=k+1}^{N_T} r_{k,i} \cdot \hat{x}_i = r_{k,k} \cdot x_k + \eta_k
  \]

- **Detection**
  \[
  \hat{x}_k = Q\{\tilde{x}_k^{IC} / r_{k,k}\}
  \]

  \[
  \text{SNR}_i = \sigma_n^{-2} |r_{i,i}|^2
  \]

- **Example for** $N_T = 4$
  \[
  \begin{pmatrix}
  \tilde{x}_1 \\
  \tilde{x}_2 \\
  \tilde{x}_3 \\
  \tilde{x}_4
  \end{pmatrix}
  =
  \begin{pmatrix}
  r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} \\
  0 & r_{2,2} & r_{2,3} & r_{2,4} \\
  0 & 0 & r_{3,3} & r_{3,4} \\
  0 & 0 & 0 & r_{4,4}
  \end{pmatrix}
  \times
  \begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
  \end{pmatrix}
  +
  \begin{pmatrix}
  \eta_1 \\
  \eta_2 \\
  \eta_3 \\
  \eta_4
  \end{pmatrix}
  \]
SIC-Detection with real MIMO-Transmission

- **Parameter**
  - **MASI**: Multiple Antenna System for ISM-Band Transmission
  - Transmission between two offices in 2.4 GHz ISM-Band
  - $N_T = N_R = 4$, 4-QAM, $\lambda/2$-ULA

### Part 4: Space-Time Signal Processing

<table>
<thead>
<tr>
<th>Layer 1</th>
<th>Layer 2</th>
<th>Layer 3</th>
<th>Layer 4</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Layer 1" /></td>
<td><img src="image2" alt="Layer 2" /></td>
<td><img src="image3" alt="Layer 3" /></td>
<td><img src="image4" alt="Layer 4" /></td>
</tr>
</tbody>
</table>

**1. Step**

**2. Step**

**3. Step**

**4. Step**

- **1. Step**: 
  - 16.9 dB

- **2. Step**: 
  - 13.7 dB

- **3. Step**: 
  - 14.6 dB

- **4. Step**: 
  - 15.5 dB
QR Decomposition with Modified Gram-Schmidt Algorithm

- QR decomposition of channel matrix: \( H = QR \)
- Gram-Schmidt

\[
\begin{bmatrix}
    h_1 & h_2 & h_3 & \cdots & h_{NT}
\end{bmatrix}
= \begin{bmatrix}
    q_1 & q_2 & q_3 & \cdots & q_{NT}
\end{bmatrix} \cdot \begin{bmatrix}
    r_{1,1} & r_{1,2} & r_{1,3} & \cdots & r_{1,nT} \\
    r_{2,1} & r_{2,2} & r_{2,3} & \cdots & r_{2,nT} \\
    r_{3,1} & r_{3,2} & r_{3,3} & \cdots & r_{3,nT} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & r_{nT,nT}
\end{bmatrix}
\]

- Example: Decomposition of \( h_3 \)
  - Columns \((q_1, q_2)\) form an orthonormal basis of the vector space \((h_1, h_2)\)
  - \(r_{1,3}\) and \(r_{2,3}\) describe the component of \( h_3 \) in the direction of \( q_1 \) and \( q_2 \)
  - \(q_3\) denotes the direction of \( h_3 \) perpendicular to the base \((q_1, q_2)\)
  - \(r_{3,3}\) describes the component of \( h_3 \) in the direction of \( q_3 \) \(\rightarrow h_3 = q_1 \cdot r_{1,3} + q_2 \cdot r_{2,3} + q_3 \cdot r_{3,3}\)

Diagonal element \(r_{k,k}\) denotes component of \(h_k\) perpendicular to base \((q_1, \ldots, q_{k-1})\)
QR Decompositions of Permutated Channel Matrices

- Given channel matrix

\[
H = \begin{bmatrix}
0.0828 & 0.3269 & 0.5548 \\
0.7662 & 0.8633 & 1.0016 \\
2.2368 & 0.6794 & 1.2594 \\
\end{bmatrix}
\]

- QR decomposition of permutated channel matrices \( H(p) \) (permutat. vector \( p \))

\[
R_{[123]} = \begin{bmatrix}
2.3659 & 0.9334 & 1.5345 \\
0 & 0.6653 & 0.7056 \\
0 & 0 & 0.2108 \\
\end{bmatrix}
\]

\[
R_{[312]} = \begin{bmatrix}
1.7021 & 2.1329 & 1.1173 \\
0 & 1.0237 & -0.1708 \\
0 & 0 & 0.1905 \\
\end{bmatrix}
\]

\[
R_{[213]} = \begin{bmatrix}
1.1462 & 1.9266 & 1.6591 \\
0 & 1.3732 & 0.3160 \\
0 & 0 & 0.2108 \\
\end{bmatrix}
\]

\[
R_{[231]} = \begin{bmatrix}
1.1462 & 1.6591 & 1.9266 \\
0 & 0.3799 & 1.1423 \\
0 & 0 & 0.7622 \\
\end{bmatrix}
\]

\[
R_{[132]} = \begin{bmatrix}
2.3659 & 1.5345 & 0.9334 \\
0 & 0.7365 & 0.6374 \\
0 & 0 & 0.1905 \\
\end{bmatrix}
\]

\[
R_{[321]} = \begin{bmatrix}
1.7021 & 1.1173 & 2.1329 \\
0 & 0.2558 & -0.6834 \\
0 & 0 & 0.7622 \\
\end{bmatrix}
\]

- Which permutation leads to best SNR in each detection step?
Adaptation of the Detection Order

- Optimization of the detection order to reduce problem of error propagation
  - Adaptation by exchanging the columns of $\mathbf{H} \rightarrow$ different QR decompositions
  - SNR$_i$ is given by diagonal element $r_{i,i}$
  - Exchange the columns of $\mathbf{H}$ in order to maximize the elements $r_{i,i}$ with respect to the detection sequence

- Sorted QR Decomposition (SQRD)
  - Optimizes the sequence within one QR Decomposition
  - Lattice determinant is independent of column sorting
  - Product of diagonal elements is constant

\[ \sqrt{\text{det} (\mathbf{H}^T \mathbf{H})} = \prod_{i=1}^{N_T} |r_{i,i}| \]

Exchange columns within the QR decomposition of $\mathbf{H}$ so that the diagonal elements $r_{i,i}$ are minimized in the sequence $r_{1,1}, r_{2,2}, \ldots$

- Small elements $r_{1,1}, r_{2,2}, \ldots$ lead to large elements $\ldots, r_{N_T - 1,N_T - 1}, r_{N_T,N_T} \rightarrow$ Post-Sorting-Algorithm

- Only very small computational effort in contrast to unsorted QRD, but does not always lead to the optimal detection sequence
Sorted QR Decomposition (SQRD)

- Modification of Gram-Schmidt algorithm by inserting a **reordering** in each decomposition step
  - Permutation vector $p$

- Decomposition step $i$
  - First $i-1$ elements of $p$ and $q_1, \ldots, q_{i-1}$ are fixed, but order of remaining columns is variable
  - Sorting rule selects column $q_{k_i}$ of remaining columns with minimum norm
  - Exchange columns of $Q$, $R$ and $p$
  - Proceed with Gram-Schmidt decomposition

\[
R = 0, \quad Q = H, \quad p = (1, \ldots, N_T) \\
\text{for } i = 1, \ldots, N_T
\]

\[k_i = \arg \min_{l=i, \ldots, N_T} \|q_l\|^2\]

exchange col. $i$ and $k_i$ in $Q$, $R$ and $p$

\[
r_{i,i} = \|q_i\|
\]

\[
q_i = q_i / r_{i,i}
\]

for $l = i + 1, \ldots, N_T$

\[
r_{i,l} = q_i^H \cdot q_l
\]

\[
q_l = q_l - r_{i,l} \cdot q_i
\]

end

end

Only one decomposition, but *optimum* sorting is not assured!
Why is heuristic sorting rule not optimal?

- Example with $N_T = 3$
  - Optimal order
    - Large part of $h_3$ perpendicular to $h_1$ and $h_2$ leads to large coefficient $r_{3,3}$
  - Suboptimal order
    - $h_2$ and $h_3$ have a large norm but similar direction, which leads to a small perpendicular component $r_{3,3} \rightarrow$ small SNR$_3$
Post Sorting Algorithm

- Relation between error covariance matrix and QR decomposition
  \[
  \Phi_{ZF} = \sigma_n^2 \cdot (H^H H)^{-1} = \sigma_n^2 \cdot (R^H Q^H QR)^{-1} = \sigma_n^2 \cdot (R^H R)^{-1} = \sigma_n^2 \cdot R^{-1} R^{-H}
  \]
  - \([\Phi_{ZF}]_{i,i}\) is proportional to the norm of the \(i\)-th row of \(R^{-1}\)
  - Due to detection order, last row of \(R^{-1}\) must have minimum norm of all rows
  - If this condition is fulfilled, the last row of the upper left \((N_T-1)\times(N_T-1)\) submatrix of \(R^{-1}\) must have minimum norm of all rows in this submatrix, ...

- Now assume, that this condition is not fulfilled for \(R^{-1}\)
  - Exchange row with minimum norm & last row \(\rightarrow\) left multiplication with \(P\)
    \(\rightarrow\) destroys triangular structure
  - Block triangular structure is achieved by multiplication with unitary Housholder matrix \(\Theta\)
    \(\rightarrow\) \(R^{-1} := R^{-1} \Theta\) and \(Q := Q \Theta\)
  - Iterate this ordering and reflection steps for upper left \((N_T-1)\times(N_T-1)\) submatrix of \(R^{-1}\), ...
Example: Efficient Sorting Algorithm for 4 Layers

- **1st iteration**
  - Row with minimum norm
  - Exchange rows
  - Block triangular structure

- **2nd iteration**
  - \( R^{-1} \)
  - \( P_1 R^{-1} \Theta_1 \)

- **3rd iteration**
  - \( P_2 P_1 R^{-1} \Theta_1 \Theta_2 \)

Mathematical expressions:

\[
R_{\text{opt}}^{-1} = P_3 P_2 P_1 R^{-1} \Theta_1 \Theta_2 \Theta_3 \\
R_{\text{opt}} = \Theta_3^H \Theta_2^H \Theta_1^H R P_1^H P_2^H P_3^H \\
Q_{\text{opt}} = Q \Theta_1 \Theta_2 \Theta_3
\]
Extension of V-BLAST to MMSE Detection (1)

- **MMSE filter matrix**
  \[
  \mathbf{G}_{\text{MMSE}} = (\mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}_{N_T})^{-1} \mathbf{H}^H
  \]

- **Output of the MMSE filter**
  \[
  \hat{\mathbf{x}}_{\text{MMSE}} = \mathbf{G}_{\text{MMSE}} \mathbf{y} = (\mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}_{N_T})^{-1} \mathbf{H}^H \mathbf{y}
  \]

- **Error covariance matrix**
  \[
  \Phi_{\text{MMSE}} = \sigma_n^2 (\mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}_{N_T})^{-1}
  \]

- **Note:** row norm of \( \mathbf{G}_{\text{MMSE}} \) does not lead to optimum sorting criterion!

- **Reason:** diagonal elements of \( \Phi_{\text{MMSE}} \) are not squared row norms of \( \mathbf{G}_{\text{MMSE}} \)

- **Compare ZF:**
  \[
  \Phi_{\text{ZF}}_{i,i} = \{(\mathbf{H}^H \mathbf{H})^{-1}\}_{i,i} = \{(\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{H} (\mathbf{H}^H \mathbf{H})^{-1}\}_{i,i} = \{(\mathbf{H}^H \mathbf{H})^{-1}\}_{i,i}
  \]
Extension of V-BLAST to MMSE Detection (2)

- Relation of MMSE to zero-forcing
  - Definition of extended channel matrix and extended receive vector
    \[
    \hat{H} = \begin{bmatrix}
    H \\
    \sigma_n I_{NT}
    \end{bmatrix} \Rightarrow \hat{H}^H \hat{H} = H^H H + \sigma_n^2 I_{NT}
    \]
  - Applying zero-forcing approach to $\hat{H}$ leads to MMSE solution w.r.t. $H$
    - Filter output expressed with $\hat{H}$ and $\hat{y}$
      \[
      \hat{x}_{\text{MMSE}} = (H^H \hat{H})^{-1} \hat{H}^H \hat{y} = H^+ \hat{y}
      \]
    - Error covariance matrix expressed with $\hat{H}$ and $\hat{y}$
      \[
      \Phi_{\text{MMSE}} = \sigma_n^2 (H^H \hat{H})^{-1} = \sigma_n^2 \cdot H^+ \hat{H}^+ H
      \]

MMSE solution corresponds to zero-forcing for extended system
MMSE-BLAST with QR Decomposition (1)

- Algorithms for ZF V-BLAST can be readily applied to MMSE V-BLAST
  - QR decomposition of extended channel matrix

\[
\mathbf{H} = \begin{bmatrix} \mathbf{H} & \sigma_n \mathbf{I}_{N_T} \end{bmatrix} = \mathbf{QR} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \mathbf{R} = \begin{bmatrix} \mathbf{Q}_1 \mathbf{R} & \mathbf{Q}_2 \mathbf{R} \end{bmatrix}
\]

with \( \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \)

\( \mathbf{Q}_1: N_R \times N_T \quad \mathbf{Q}_2: N_T \times N_T \)

- Attention: \( \mathbf{Q}_1 \) and \( \mathbf{Q}_2 \) are not unitary since they contain only column parts of \( \mathbf{Q} \! \)

- No matrix inversion for efficient optimum sorting algorithm required

\( \sigma_n \mathbf{I}_{N_T} = \mathbf{Q}_2 \mathbf{R} \Rightarrow \mathbf{R}^{-1} = \sigma_n^{-1} \mathbf{Q}_2 \)

- Compensates for higher computational effort of QR decomposition

- Filtered receive signal

\[
\tilde{\mathbf{x}} = \mathbf{Q}^H \mathbf{y} = \begin{bmatrix} \mathbf{Q}_1^H & \mathbf{Q}_2^H \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ 0 \end{bmatrix} = \mathbf{Q}_1^H \mathbf{y} = \mathbf{Q}_1^H (\mathbf{Hx} + \mathbf{n}) = \mathbf{Q}_1^H \mathbf{Hx} + \mathbf{Q}_1^H \mathbf{n}
\]

- Analyzing \( \mathbf{Q}_1^H \mathbf{H} \)

\[
\mathbf{Q}^H \mathbf{H} = \mathbf{R} = \begin{bmatrix} \mathbf{Q}_1^H & \mathbf{Q}_2^H \end{bmatrix} \mathbf{H} = \mathbf{Q}_1^H \mathbf{H} + \sigma_n \mathbf{Q}_2^H
\]
MMSE-BLAST with QR Decomposition

- Extracting $\mathbf{R}$
  \[ \mathbf{Q}^H \mathbf{H} = \mathbf{R} = \begin{bmatrix} \mathbf{Q}_1^H & \mathbf{Q}_2^H \end{bmatrix} \mathbf{H} = \mathbf{Q}_1^H \mathbf{H} + \sigma_n \mathbf{Q}_2^H \]

- Inserting above result into filtered receive signal
  \[ \mathbf{x} = \mathbf{Q}_1^H \mathbf{H} \mathbf{x} + \mathbf{Q}_1^H \mathbf{n} = \mathbf{R} \mathbf{x} - \sigma_n \mathbf{R}^{-H} \mathbf{x} + \mathbf{Q}_1^H \mathbf{n} \]

Perform successive interference cancellation like before

Second term represents remaining interference
Extension for Quadrature Amplitude Modulation

- QAM symbols → real and imaginary parts are independent of each other
  - Real-valued system model
    \[ \begin{bmatrix} y_r \\ y_i \end{bmatrix} = \begin{bmatrix} H_r & -H_i \\ H_i & H_r \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} + \begin{bmatrix} n_r \\ n_i \end{bmatrix} \]

- Real system with doubled number of transmit and receive antennas
  - One QAM symbol does not need to be detected completely
  - Increased degrees of freedom for finding the optimum detection order
  - Leads to additional performance gain

- All algorithms described before can be used without modification
  - Only real-valued operations necessary
  - Nevertheless, slightly increased computational complexity due to larger matrices
Bit Error Rates for V-BLAST Systems

- Simulation parameters: $N_T = N_R = 4$, QPSK $\rightarrow$ 8 bits per time instant
- Enormous performance gain by sorting $\rightarrow$ SQRD close to optimum for ZF
  For MMSE SQRD is near optimum only for low SNR

![Graph showing BER vs Eb/N0 for different methods: linear, QRD-SIC, SQRD-SIC, V-BLAST, Genie SIC, MLD.](image)

**Part 4: Space-Time Signal Processing**
Bit Error Rates for QAM Extension of MMSE V-BLAST

Simulation parameters:
- 4 transmit antennas
- 4 receive antennas
- Flat Rayleigh fading
- Uncorrelated channels
- QPSK modulation
- Uncoded data streams

Result:
- Small performance gain without sorting
- Up to 2dB gain with optimum ordering

**Graph:**
- BER vs. $E_b/N_0$ in dB
- Curves for unsorted, SQRD, and optimum ordering
- Y-axis: BER
- X-axis: $E_b/N_0$ in dB

---

**Part 4: Space-Time Signal Processing**
Analysis of Error Propagation

- Uncoded system with $N_T = N_R = 4$ antennas
- For genie detection the diversity of layer $i$ is given by $N_R - i + 1$
Decision Regions of the Detection Schemes ($N_T = 2$)

- Maximum-Likelihood (ML): Voronoi regions (nearest neighbor)
- Linear Detection (LD): parallelogram in direction of $h_1$ and $h_2$
- Successive Interference Cancellation (SIC): rectangle in direction of $q_2$ and $q_1$

- QR decomposition

$$\begin{bmatrix} h_1 & h_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \cdot \begin{bmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{bmatrix}$$
Basic Principle of Sphere Detection (1)

- Maximum-Likelihood Criterion:
  \[ \hat{x}_{ML} = \arg\min_{x' \in S} \| y - Hx' \|_2 \]

- Basic idea of Sphere Detection (SD):
  - Restrict the search to hypothesis \( x' \) within ball of radius \( d_{r'}^2 \) around \( y \) → easy?
  \[ d_{r'}^2 \geq \| y - Hx' \|_2^2 = \| y - Q \overline{R} x' \|_2^2 \]

  - Multiplication with \( Q^T \) (orthogonal matrix) does not change distance
  \[ d_{r'}^2 \geq \| Q^T y - Q^T Q \overline{R} x' \|_2^2 = \| Q^T y - R x' \|_2^2 + \| Q_y \|_2^2 = \| \tilde{x} - R x' \|_2^2 + \| Q_y \|_2^2 \]

  - Last term is independent of the hypothesis → define radius
  \[ d_r^2 = d_{r'}^2 - \| Q_y \|_2^2 \]

- Upper triangular form of \( R \) → successive testing of hypothesis (compare SIC)

Equivalence real-valued system model is assumed in the sequel!
Basic Principle of Sphere Detection (2)

- 1. Step
  - Simplify constraint
    \[ d_r^2 \geq (\tilde{x}_{N_{2T}} - r_{N_{2T}}, N_{2T} x'_{N_{2T}})^2 + \sum_{i=1}^{N_{2T}-1} (\tilde{x}_i - \sum_{\nu=i}^{N_{2T}} r_{i,\nu} x'_{\nu})^2 \]
  - Choose hypothesis \( \hat{x}'_{N_{2T}} \) that fulfills
  - Update the constraint for the remaining layers
  \[ \Delta_{N_{2T}-1}^2 = \Delta_{N_{2T}}^2 - \delta_{N_{2T}}^2 \]
  \[ \delta_{N_{2T}}^2 = (\tilde{x}_{N_{2T}} - r_{N_{2T}}, N_{2T} \hat{x}'_{N_{2T}})^2 \]

- 2. Step
  - Choose hypothesis \( \hat{x}'_{N_{2T}-1} \) that fulfills
  \[ \Delta_{N_{2T}-1}^2 \geq (\tilde{x}_{N_{2T}-1} - r_{N_{2T}-1}, N_{2T}-1 \hat{x}'_{N_{2T}-1} + r_{N_{2T}-1}, N_{2T} \hat{x}'_{N_{2T}})^2 \]
  - Update the constraint for the remaining layers
  \[ \Delta_{N_{2T}-1}^2 = \Delta_{N_{2T}-1}^2 - \delta_{N_{2T}-1}^2 \]
  \[ \delta_{N_{2T}-1}^2 = (\tilde{x}_{N_{2T}-1} - r_{N_{2T}-1}, N_{2T}-1 \hat{x}'_{N_{2T}-1} + r_{N_{2T}-1}, N_{2T} \hat{x}'_{N_{2T}})^2 \]

...
Search Strategies within each layer

- **Fincke-Pohst (FP-SD)**
  - Determine the range of allowed values for
  - Choose symbols in ascending order
  - Radius $d_r$ must be initialized appropriately
  - After a valid estimation is found, $d_r$ is reduced and new search is started from root

- **Schnorr-Euchner (SE-SD)**
  - Consider symbols close to the interference reduced signal first
  - Initialization $d_r = \infty \rightarrow$ first point found corresponds to SIC result (Babai-Point)!
  - Update radius if a new point is found and continue search in layer 2, 3, …
Searching Tree of Schnorr-Euchner for $N_{2T}^2=4$, 2-ASK per real layer

$\hat{x}_{ML} = [+1 +1 -1 +1]$
Some Aspects of Implementation

- Computational complexity is determined by number of visited nodes
- Optimization of the detection order
  - Due to the tree structure, a good estimation of hypothesis in first steps is desired → efficient search if the first point is as close to ML as possible
  - By application of SQRD an optimized sorting is achieved
- Choice of initial radius
  - SE-SD selects in each layer the nearest hypothesis → for $d_r=\infty$ → $\hat{x} = \hat{x}_{SIC}$
    Nevertheless, an adequate choice of $d_r$ leads to an advance of speeding
  - FP-SD requires a suitable choice of $d_r$ → arrange $d_r$ due to noise, e.g. Hassibi:
    $$
    \|y - H\hat{x}_{ML}\|^2 = \|n\|^2 \sim \chi^2_{N_{2R}}
    $$
    $$
    d_{r'} = \alpha E\{\|n\|^2\} = \alpha \sigma_n^2 N_R
    $$
    $$
    P_{FP} = \frac{1}{\Gamma(N_R)\gamma(N_R, \alpha N_R)} = 1 - \epsilon
    $$
- Extension to MMSE criterion
  - For SIC the MMSE-extension leads to an improved estimate → speeding up
  - Solution is not necessary ML-solution due to the ignored interference
Complexity Evaluation

- Impact of initial radius and sorting for $N_T = N_R = 4$, 16-QAM ($\rightarrow 65536$ points) to the average number of visited nodes (Hassibi with $P_{FP}=0.99$)
- SQRD and MMSE lead to strong decrease in complexity for both schemes
- SE-SD is first choice of ML-Implementation (notice $d_r=1000$ !)

![Graphs showing complexity evaluation for two different algorithms with varying $E_b/N_0$ in dB]
Performance of MMSE Sphere Detection

- **Simulation parameters**
  - $N_T = N_R = 4$, 16-QAM

- **Results**
  - Small performance loss of MMSE-extension due to ignored interference term
  - Complexity is significantly reduced by MMSE-extension
    → first choice of implementation
Hardware Implementation

- Real-time implementation of SQRD & Co. at ETH Zürich:
  http://www.cc.ethz.ch/media/picturelibrary/archiv/mimotechnik/index
Selected References for V-BLAST Systems

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Part 4: Space-Time Signal Processing