# BER-based Power Allocation for Decode-and-Forward Relaying with $M$-QAM Constellations

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***Abstract*—In this paper we develop a power allocation scheme for single-relay systems applying Decode-and-Forward (DF) based on the resulting bit error rate (BER) at the destination. First, an analytical expression for the BER of $M$-QAM modulation considering estimation errors at the relay is derived. Based on this expression, the total transmit power is optimally assigned to the source and the relay in order to minimize the probability of errors at the destination. The preciseness of the derived closed form expression as well as the superior performance of the proposed DF-based relaying system are demonstrated by simulation results.**

## I. Introduction

Nowadays cooperative relaying systems attract much attention due to the possible enhancements of the end-to-end performance by introducing relay nodes. In the basic triplet relaying system the communication from source to destination is supported by one relay. For Amplify-and-Forward (AF) the relay amplifies the signal received from the source and forwards it to the destination or it estimates the transmitted message and forwards this quantized symbols (Decode-and-Forward, DF). In contrast to assigning equal transmit power to the source and the relay, an adapted power allocation can further improve the system performance. Power allocation schemes have been proposed to maximize the system capacity for AF and DF, e.g., in [1], [2]. A BER-based power allocation scheme to minimize the BER at the destination in case of AF relaying has been proposed in [3] based on the resulting signal-to-noise-ratio (SNR) after maximum-ratio-combining (MRC). Contrarily, the derivation of analytical BER expressions in case of DF is not straightforward as the regeneration of the symbols at the relay is a non-linear operation, resulting in more involved expressions for power allocation. In [4], [5] the BER expression at the destination is derived for adaptive DF, where the relay only transmits when the message is correctly decided. Considering the effect of error propagation at the relay, the BER of common DF relaying for BPSK modulation is presented in [6], [7]. These works have been generalized to $M$-QAM without the direct link from source to destination in [8]. In [9] the authors presented a power allocation scheme based on an approximated BER expression for uncoded and coded triplet relaying systems assuming AWGN channels and $M$-QAM modulation. In this paper we extend the derivation by a more accurate closed-form BER expression for $M$-QAM and DF with error propagation at the relay considering the direct transmission and fading channels as well. Based on the derived BER expression an optimized power allocation is addressed leading to superior performance.

The remainder of the paper is organized as follows. The system model is introduced in Section II. In Section III the closed-form BER expression for DF is derived for both BPSK and $M$-QAM. The optimum assignment of the total power in order to minimize the BER at the destination is presented in Section IV. The simulation results in Section V verify the quality of the derivations and the proposed power allocation scheme. Section VI concludes this paper.

## II. System Description

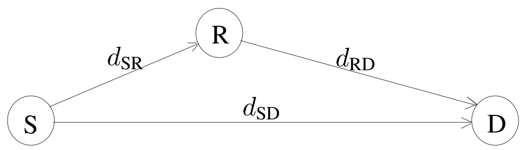


Fig. 1. A triplet relaying system model consisting of source S, relay R and destination D. The distances of the SR, RD and SD links are denoted as $d_{\mathrm{SR}}$, $d_{\mathrm{RD}}$ and $d_{\mathrm{SD}}$, respectively.

We consider the uncoded triplet relaying system shown in Fig. 1, where the relay R supports the communication from source S to destination D. Based on the half-duplex assumption, S broadcasts the source message $x_{\mathrm{S}}$ with power $\mathcal{P}_{\mathrm{S}}$ and $\mathrm{E}\{|x_{\mathrm{S}}|^2\} = 1$ in the first time slot to both R and D. On receiving the message $y_{\mathrm{SR}}$ from the source, R constructs the message $x_{\mathrm{R}}$ with power $\mathcal{P}_{\mathrm{R}}$ and $\mathrm{E}\{|x_{\mathrm{R}}|^2\} = 1$ using either AF or DF and forwards it to D in the second time slot. The three different receive signals are given by

$$y_{\mathrm{SR}} = \sqrt{\mathcal{P}_{\mathrm{S}}} h_{\mathrm{SR}} x_{\mathrm{S}} + n_{\mathrm{SR}} \tag{1a}$$

$$y_{\mathrm{SD}} = \sqrt{\mathcal{P}_{\mathrm{S}}} h_{\mathrm{SD}} x_{\mathrm{S}} + n_{\mathrm{SD}} \tag{1b}$$

$$y_{\mathrm{RD}} = \sqrt{\mathcal{P}_{\mathrm{R}}} h_{\mathrm{RD}} x_{\mathrm{R}} + n_{\mathrm{RD}} \ , \tag{1c}$$

where $n_{\text{SR}}$, $n_{\text{SD}}$ and $n_{\text{RD}}$ represent additive white Gaussian noise (AWGN) terms with variance $\sigma_n^2$. The coefficients $h_{\text{SR}}$, $h_{\text{SD}}$ and $h_{\text{RD}}$ contain the path-loss effects as well as fading. If the distance from source to destination is fixed to $d_{\text{SD}} = 1$, the transmission gains can be modeled by

$$h_{\text{SR}} = f_{\text{SR}} \left( \frac{1}{d_{\text{SR}}} \right)^{\frac{\alpha}{2}}, \quad h_{\text{RD}} = f_{\text{RD}} \left( \frac{1}{d_{\text{RD}}} \right)^{\frac{\alpha}{2}} \text{ and } \quad h_{\text{SD}} = f_{\text{SD}} \ , \tag{2}$$

where $f_{\text{SR}}$, $f_{\text{SD}}$ and $f_{\text{RD}}$ are the fading coefficients and $\alpha$ denotes the path-loss exponent. The SNRs of the different links are given by

$$\gamma_{\text{SR}} = \frac{\mathcal{P}_{\text{S}}|h_{\text{SR}}|^2}{\sigma_n^2}, \quad \gamma_{\text{SD}} = \frac{\mathcal{P}_{\text{S}}|h_{\text{SD}}|^2}{\sigma_n^2} \text{ and } \gamma_{\text{RD}} = \frac{\mathcal{P}_{\text{R}}|h_{\text{RD}}|^2}{\sigma_n^2} \ . \tag{3}$$

Using the weighting factors $w_{\text{SD}}$ and $w_{\text{RD}}$ the destination linearly combines the receive signals $y_{\text{SD}}$ and $y_{\text{RD}}$ to achieve the estimate

$$y_{\text{D}} = w_{\text{SD}} y_{\text{SD}} + w_{\text{RD}} y_{\text{RD}} \tag{4}$$

for the source signal. In case of AF and MRC the optimal SNR at the destination equals [1]

$$\gamma_{\text{D}} = \frac{\gamma_{\text{SR}}\gamma_{\text{RD}}}{\gamma_{\text{SR}} + \gamma_{\text{RD}} + 1} + \gamma_{\text{SD}} \ . \tag{5}$$

The probability of an error event $e_{\text{D}}$ at D in case of BPSK is given by [10]

$$P_b(e_{\text{D}}) = \frac{1}{2}\text{erfc}\left(\sqrt{\gamma_{\text{D}}}\right) \tag{6}$$

and a corresponding approximation for $M$-QAM with Gray mapping reads as

$$P_b(e_{\text{D}}) \approx \frac{2}{\log_2 M}\left(1 - \frac{1}{\sqrt{M}}\right) \text{erfc}\left(\sqrt{\frac{3}{2(M-1)}\gamma_{\text{D}}}\right) \ . \tag{7}$$

In order to minimize the BER at D, the total transmit power $\mathcal{P}_{\text{tot}} = \mathcal{P}_{\text{S}} + \mathcal{P}_{\text{R}}$ has to be allocated to S and R such that the SNR $\gamma_{\text{D}}$ is maximized.

In case of DF one may apply MRC combining with weighting factors $w_{\text{SD}} = h_{\text{SD}}^*\sqrt{\mathcal{P}_{\text{S}}}$ and $w_{\text{RD}} = h_{\text{RD}}^*\sqrt{\mathcal{P}_{\text{R}}}$ yielding the combiner output signal

$$\begin{aligned} y_{\text{D}} =& h_{\text{SD}}^*\sqrt{\mathcal{P}_{\text{S}}} y_{\text{SD}} + h_{\text{RD}}^*\sqrt{\mathcal{P}_{\text{R}}} y_{\text{RD}} \\ =& |h_{\text{SD}}|^2\mathcal{P}_{\text{S}} x_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}} x_{\text{R}} + \\ & h_{\text{SD}}^*\sqrt{\mathcal{P}_{\text{S}}} n_{\text{SD}} + h_{\text{RD}}^*\sqrt{\mathcal{P}_{\text{R}}} n_{\text{RD}} \\ =& |h_{\text{SD}}|^2\mathcal{P}_{\text{S}} x_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}} x_{\text{R}} + \tilde{n} \ , \end{aligned} \tag{8}$$

where $\tilde{n}$ is the equivalent noise with variance $\sigma_{\tilde{n}}^2 = \left(|w_{\text{SD}}|^2 + |w_{\text{RD}}|^2\right)\sigma_n^2$. Subsequently, (8) is used to derive appropriate expressions for the BER at D and develop corresponding power allocation schemes. However, reduced error rates may also be achieved by other choices for the weighting factors as discussed in [7], [11].

## III. Closed-form BER Analysis for DF

For DF, the relay estimates the source signal $x_{\text{S}}$ using the receive signal $y_{\text{SR}}$, which may, or may not, introduce errors. Obviously, an error event $e_{\text{R}}$ and a non-error event $\overline{e}_{\text{R}}$ at R will affect the data recovery at D differently, and should accordingly be considered separately. By denoting the probability of an error event $e_{\text{R}}$ at R by $\text{P}(e_{\text{R}})$ and indicating its complement, i.e., the probability of a correct decision, by $\text{P}(\overline{e}_{\text{R}}) = 1 - \text{P}(e_{\text{R}})$ the error probability at D corresponds to [6]

$$\text{P}(e_{\text{D}}) = \text{P}(\overline{e}_{\text{R}})\text{P}(e_{\text{D}}|\overline{e}_{\text{R}}) + \text{P}(e_{\text{R}})\text{P}(e_{\text{D}}|e_{\text{R}}) \ . \tag{9}$$

Here, $\text{P}(e_{\text{D}}|e_{\text{R}})$ represent the error probability at D conditioned by an error event at R. Similarly, $\text{P}(e_{\text{D}}|\overline{e}_{\text{R}})$ is the probability of an error at D in case that the estimation at R was correct. Subsequently, closed-form expressions will be derived for (9) for BPSK and $M$-QAM.

### A. BPSK Modulation

Inserting the SNR $\gamma_{\text{SR}}$ in (6) the error probability at R for BPSK is given by

$$\text{P}(e_{\text{R}}) = \frac{1}{2}\text{erfc}\left(\frac{|h_{\text{SR}}|\sqrt{\mathcal{P}_{\text{S}}}}{\sigma_n}\right) \ . \tag{10}$$

In case of a correct decision at R (i.e., event $\overline{e}_{\text{R}}$), $x_{\text{R}} = x_{\text{S}}$ holds and thus (8) becomes

$$y_{\text{D}} = \left(|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}\right) x_{\text{S}} + \tilde{n} \ , \tag{11}$$

which leads to the error probability

$$\text{P}(e_{\text{D}}|\overline{e}_{\text{R}}) = \frac{1}{2}\text{erfc}\left(\frac{|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}}{\sigma_{\tilde{n}}}\right) \ . \tag{12}$$

In contrast, an erroneous estimation at R (i.e., event $e_{\text{R}}$) results in $x_{\text{R}} = -x_{\text{S}}$. Consequently, the combiner output (8) equals

$$y_{\text{D}} = \left(|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} - |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}\right) x_{\text{S}} + \tilde{n} \tag{13}$$

and describes the effect of error propagation in DF as the received signals are destructively combined. The corresponding error probability is

$$\text{P}(e_{\text{D}}|e_{\text{R}}) = \frac{1}{2}\text{erfc}\left(\frac{|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} - |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}}{\sigma_{\tilde{n}}}\right) \ . \tag{14}$$

Using (10), (12) and (14) in (9) the error probability at D for BPSK is

$$\begin{aligned} \text{P}(e_{\text{D}}) =& \left(1 - \frac{1}{2}\text{erfc}\left(|h_{\text{SR}}|\frac{\sqrt{\mathcal{P}_{\text{S}}}}{\sigma_n}\right)\right) \cdot \\ & \frac{1}{2}\text{erfc}\left(\frac{|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}}{\sigma_{\tilde{n}}}\right) + \\ & \frac{1}{4}\text{erfc}\left(|h_{\text{SR}}|\frac{\sqrt{\mathcal{P}_{\text{S}}}}{\sigma_n}\right) \text{erfc}\left(\frac{|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} - |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}}{\sigma_{\tilde{n}}}\right) \ . \end{aligned} \tag{15}$$

and equals the BER $P_b$. Note that the second term in (15) incorporates the effect of error propagation by R on the overall BER performance.

### B. $M$-QAM Modulation

In order to calculate the error probability for $M$-QAM we can also investigate the error probability of the corresponding $\sqrt{M}$-ASK modulation, as the real and the imaginary parts of the complex constellation are independent to each other. Furthermore, to simplify the derivation it is assumed that an error event either at R or at D only causes neighboring symbol errors.

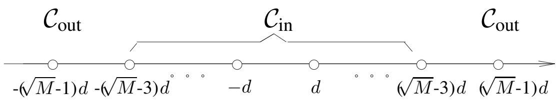


Fig. 2. Symbol constellation for $\sqrt{M}$-ASK

Unlike BPSK, a symbol in the constellation is not only determined by its sign but also by its amplitude for $\sqrt{M}$-ASK, which requires different treatment for 'inner' symbols and 'outer' symbols. As displayed in Fig. 2, the set $\mathcal{C}_{\text{in}}$ contains the inner symbols $x_{\text{S}} = \pm d, \pm 3d, \cdots, \pm(\sqrt{M}-3)d$ and its complementary set $\mathcal{C}_{\text{out}}$ contains $x_{\text{S}} = \pm(\sqrt{M}-1)d$, where $d$ is half of the minimum Euclidean distance in the constellation given by

$$d = \sqrt{\frac{3}{2(M-1)}} \ . \tag{16}$$

Distinguishing the different symbol sets in the equation for the error probability (9) leads to

$$\begin{aligned} \text{P}(e_{\text{D}}) =& \text{P}\left(e_{\text{D}}|\overline{e}_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \text{P}\left(\overline{e}_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) + \\ & \text{P}\left(e_{\text{D}}|\overline{e}_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \text{P}\left(\overline{e}_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) + \\ & \text{P}\left(e_{\text{D}}|e_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \text{P}\left(e_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) + \\ & \text{P}\left(e_{\text{D}}|e_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \text{P}\left(e_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \ . \end{aligned} \tag{17}$$

As the inner and the outer symbols occur with probabilities, i.e., $\text{P}\left(x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) = \frac{\sqrt{M}-2}{\sqrt{M}}$ and $\text{P}\left(x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) = \frac{2}{\sqrt{M}}$, (17) has to be extended to

$$\begin{aligned} \text{P}(e_{\text{D}}) = & \text{P}\left(e_{\text{D}}|\overline{e}_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \text{P}\left(\overline{e}_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \text{P}\left(x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \\ & + \text{P}\left(e_{\text{D}}|\overline{e}_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \text{P}\left(\overline{e}_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \text{P}\left(x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \\ & + \text{P}\left(e_{\text{D}}|e_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \text{P}\left(e_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \text{P}\left(x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \\ & + \text{P}\left(e_{\text{D}}|e_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \text{P}\left(e_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \text{P}\left(x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \ . \end{aligned} \tag{18}$$

To calculate $\text{P}(e_{\text{D}})$ in (18), the symbol error probability at R and its complementary probability subject to different transmit symbols are derived and given by [10]

$$\text{P}\left(e_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) = \frac{1}{2}\text{erfc}\left(\frac{|h_{\text{SR}}|\sqrt{\mathcal{P}_{\text{S}}}d}{\sqrt{2}\sigma_n}\right) \tag{19a}$$

$$\text{P}\left(e_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) = \text{erfc}\left(\frac{|h_{\text{SR}}|\sqrt{\mathcal{P}_{\text{S}}}d}{\sqrt{2}\sigma_n}\right) \tag{19b}$$

$$\text{P}\left(\overline{e}_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) = 1 - \text{P}\left(e_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{out}}\right) \tag{19c}$$

$$\text{P}\left(\overline{e}_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) = 1 - \text{P}\left(e_{\text{R}}|x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) \tag{19d}$$

for $x_{\text{S}} \in \mathcal{C}_{\text{out}}$ and $x_{\text{S}} \in \mathcal{C}_{\text{in}}$, respectively.

A correct decision at R leads to $x_{\text{R}} = x_{\text{S}}$. Therefore, the receive signal at D after MRC can be expressed as that in (11), which leads to the following probability expressions

$$\text{P}(e_{\text{D}}|\overline{e}_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{out}}) = \frac{1}{2}\text{erfc}\left(\frac{|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}}{\sigma_{\tilde{n}}}\right) \tag{20a}$$

$$\text{P}(e_{\text{D}}|\overline{e}_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{in}}) = \text{erfc}\left(\frac{|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}}{\sigma_{\tilde{n}}}\right) \ . \tag{20b}$$

To derive the symbol error probability terms at D conditioned by an error event $e_{\text{R}}$ at R, we need to investigate the effect of the error event on the probability density function of the receive signal at D. Since an error event at R is only assumed for neighboring symbols, $x_{\text{R}}$ resulting from an error occurrence can be written as

$$x_{\text{R}} = \begin{cases} x_{\text{S}} \pm 2d & \text{if } x_{\text{S}} \in \mathcal{C}_{\text{in}} \ , \\ x_{\text{S}} + 2d & \text{if } x_{\text{S}} = -(\sqrt{M}-1)d \ , \\ x_{\text{S}} - 2d & \text{if } x_{\text{S}} = (\sqrt{M}-1)d \end{cases} \tag{21}$$

with respect to different transmit symbols $x_{\text{S}}$.

For the first case in (21), the receive signal after combining using $w_{\text{SD}}$ and $w_{\text{RD}}$ at D can be re-written as

$$y_{\text{D}} = \left(|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}\right) x_{\text{S}} \pm 2d\,|h_{\text{RD}}|^2\mathcal{P}_{\text{R}} + \tilde{n} \ , \tag{22}$$

where it is supposed that the probability of a left-side error (-) is equal to that of a right-side error (+). This enables us to focus on one side only due to the symmetry property, e.g., $x_{\text{R}} = x_{\text{S}} + 2d$. Therefore, the following probability expression holds

$$\begin{aligned} & \text{P}\left(e_{\text{D}}|e_{\text{R}}, x_{\text{S}} \in \mathcal{C}_{\text{in}}\right) = \\ & \int_{-\infty}^{\tilde{h}(x_{\text{S}}-d)} \frac{1}{\sqrt{2\pi\sigma_{\tilde{n}}^2}} \exp\left[-\frac{\left(t - \tilde{h}x_{\text{S}} - 2d\,|h_{\text{RD}}|^2\mathcal{P}_{\text{R}}\right)^2}{2\sigma_{\tilde{n}}^2}\right] dt + \\ & \int_{\tilde{h}(x_{\text{S}}+d)}^{\infty} \frac{1}{\sqrt{2\pi\sigma_{\tilde{n}}^2}} \exp\left[-\frac{\left(t - \tilde{h}x_{\text{S}} - 2d\,|h_{\text{RD}}|^2\mathcal{P}_{\text{R}}\right)^2}{2\sigma_{\tilde{n}}^2}\right] dt \\ & = \frac{1}{2}\text{erfc}\left(\frac{\left(|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + 3|h_{\text{RD}}|^2\mathcal{P}_{\text{R}}\right)d}{\sqrt{2}\sigma_{\tilde{n}}}\right) + \\ & \quad \frac{1}{2}\text{erfc}\left(\frac{\left(|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} - |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}\right)d}{\sqrt{2}\sigma_{\tilde{n}}}\right) \ , \end{aligned} \tag{23}$$

with $\tilde{h} = |h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}$ for simplicity. For the second case in (21), $y_{\text{D}}$ reads

$$y_{\text{D}} = \left(|h_{\text{SD}}|^2\mathcal{P}_{\text{S}} + |h_{\text{RD}}|^2\mathcal{P}_{\text{R}}\right) x_{\text{S}} + 2d\,|h_{\text{RD}}|^2\mathcal{P}_{\text{R}} + \tilde{n} \ , \tag{24}$$

where $x_{\text{S}} = -(\sqrt{M}-1)d$. Since $x_{\text{S}}$ is the left boundary constellation point, the corresponding error probability contains only one integral term and is calculated by

$$\mathrm{P}\left(e_{\mathrm{D}}|e_{\mathrm{R}},x_{\mathrm{S}}=-(\sqrt{M}-1)d\right)=$$
$$\int\limits_{\tilde{h}(x_{\mathrm{S}}+d)}^{\infty}\frac{1}{\sqrt{2\pi\sigma_{\tilde{n}}^{2}}}\exp\left[-\frac{\left(t-\tilde{h}x_{\mathrm{S}}-2d\,|h_{\mathrm{RD}}|^{2}\mathcal{P}_{\mathrm{R}}\right)^{2}}{2\sigma_{\tilde{n}}^{2}}\right]dt \tag{25}$$
$$=\frac{1}{2}\mathrm{erfc}\left(\frac{\left(|h_{\mathrm{SD}}|^{2}\mathcal{P}_{\mathrm{S}}-|h_{\mathrm{RD}}|^{2}\mathcal{P}_{\mathrm{R}}\right)d}{\sqrt{2}\sigma_{\tilde{n}}}\right).$$

For the third case in (21), we deduce

$$\mathrm{P}\left(e_{\mathrm{D}}|e_{\mathrm{R}},x_{\mathrm{S}}=(\sqrt{M}-1)d\right)=\mathrm{P}\left(e_{\mathrm{D}}|e_{\mathrm{R}},x_{\mathrm{S}}=-(\sqrt{M}-1)d\right)$$

due to symmetry property yielding

$$\mathrm{P}\left(e_{\mathrm{D}}|e_{\mathrm{R}},x_{\mathrm{S}}\in\mathcal{C}_{\mathrm{out}}\right)=\frac{1}{2}\mathrm{erfc}\left(\frac{\left(|h_{\mathrm{SD}}|^{2}\mathcal{P}_{\mathrm{S}}-|h_{\mathrm{RD}}|^{2}\mathcal{P}_{\mathrm{R}}\right)d}{\sqrt{2}\sigma_{\tilde{n}}}\right). \tag{26}$$

Finally, by inserting the probabilities (19), (20), (23), and (26) into (18) the closed-form expression for the symbol error probability for $M$-QAM at D

$$\begin{aligned}\mathrm{P}(e_{\mathrm{D}})=&\frac{\sqrt{M}-1}{\sqrt{M}}\mathrm{erfc}\left(\frac{(|h_{\mathrm{SD}}|^{2}\mathcal{P}_{\mathrm{S}}+|h_{\mathrm{RD}}|^{2}\mathcal{P}_{\mathrm{R}})d}{\sqrt{2}\sigma_{\tilde{n}}}\right)\\&+\mathrm{erfc}\left(\frac{|h_{\mathrm{SR}}|d}{\sqrt{2}\sigma_{n}}\right)\cdot\\&\left[\frac{2\sqrt{M}-3}{2\sqrt{M}}\mathrm{erfc}\left(\frac{(|h_{\mathrm{SD}}|^{2}\mathcal{P}_{\mathrm{S}}+|h_{\mathrm{RD}}|^{2}\mathcal{P}_{\mathrm{R}})d}{\sqrt{2}\sigma_{\tilde{n}}}\right)\right.\\&+\frac{\sqrt{M}-1}{2\sqrt{M}}\mathrm{erfc}\left(\frac{(|h_{\mathrm{SD}}|^{2}\mathcal{P}_{\mathrm{S}}-|h_{\mathrm{RD}}|^{2}\mathcal{P}_{\mathrm{R}})d}{\sqrt{2}\sigma_{\tilde{n}}}\right)\\&\left.+\frac{\sqrt{M}-2}{2\sqrt{M}}\mathrm{erfc}\left(\frac{(|h_{\mathrm{SD}}|^{2}\mathcal{P}_{\mathrm{S}}+3|h_{\mathrm{RD}}|^{2}\mathcal{P}_{\mathrm{R}})d}{\sqrt{2}\sigma_{\tilde{n}}}\right)\right]\end{aligned} \tag{27}$$

is achieved. This yields the following approximation for the bit error probability at D for $\sqrt{M}$-ASK as

$$P_b=\frac{2}{\log_2 M}\mathrm{P}(e_{\mathrm{D}}) \tag{28}$$

by utilizing Gray mapping, which is equivalent to that of $M$-QAM when $\frac{E_b}{N_0}$ is used as a measurement of SNR [10].

## IV. BER-based Power Allocation

By exploiting the derived closed-form BER expressions for BPSK in (15) and $M$-QAM in (28) power allocation can be performed in order to minimize the BER at D under a total power constraint $\mathcal{P}_{\mathrm{tot}}=\mathcal{P}_{\mathrm{S}}+\mathcal{P}_{\mathrm{R}}$ leading to the optimization problem

$$\min.\ P_b(e_{\mathrm{D}}) \tag{29a}$$
$$\text{s.t. } \mathcal{P}_{\mathrm{tot}}=\mathcal{P}_{\mathrm{S}}+\mathcal{P}_{\mathrm{R}}\,. \tag{29b}$$

The power constraint can be incorporated into the optimization by defining $\mathcal{P}_{\mathrm{S}}=\rho\mathcal{P}_{\mathrm{tot}}$ and $\mathcal{P}_{\mathrm{R}}=(1-\rho)\mathcal{P}_{\mathrm{tot}}$, where $\rho$ is the proportion of power allocated to S and $0\leq\rho\leq 1$.

However, the solution to the optimization problem above is not straightforward due to the rather complicated expressions in (15) and (28). Thus we resort to a numerical search for the optimized value of $\rho$ that minimizes the corresponding BER. Note, that this is a centralized power allocation scheme and the central unit requires the instantaneous channel knowledges of all the links.

## V. Simulation Results

In the simulation setups, it is assumed that R joins the direct line between S and D with $0\leq d_{\mathrm{SR}}\leq 1$. The path-loss exponent is set to $\alpha=4$ and the total available power is set to $\mathcal{P}_{\mathrm{tot}}=0\,\mathrm{dBm}$. First, AWGN scenarios with fixed coefficients $f_{\mathrm{SD}}=f_{\mathrm{SR}}=f_{\mathrm{RD}}=1$ are considered. In combination with the used path-loss model (2) the normalization with respect to $d_{\mathrm{SD}}=1$ leads to amplifications on the SR and RD links. Later on, Rayleigh fading channels are considered. The SNR is given by $E_b/N_0=\mathcal{P}_{\mathrm{tot}}/(\sigma_n^2\log_2 M)$.

To verify the preciseness of derived BER expressions at D, the probability expressions in (15), (28) for DF and (6), (7) for AF as well as the results achieved by simulations for BPSK, 4-QAM, 16-QAM and 64-QAM are shown in Fig. 3 with distance $d_{\mathrm{SR}}=0.5$ and without power allocation ($\rho=0.5$). As can be observed, the analytical results for both DF and AF correspond to the simulation with great precision for all modulation schemes under investigation. Note that BPSK and 4-QAM share the same BER behavior with respect to $E_b/N_0$.

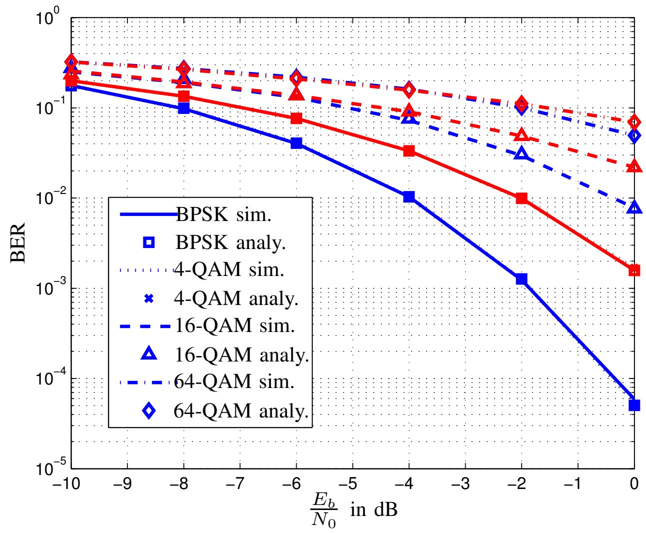


Fig. 3. BER for BPSK, 4-QAM, 16-QAM and 64-QAM from simulation and analysis with $d_{\mathrm{SR}}=0.5$ and $\rho=0.5$ (without power allocation), AWGN channels, blue curves for DF and red curves for AF.

For different relay positions Fig. 4 shows the proportion $\rho$ of power allocated to S for DF achieved by (29) and for AF using the result from [3]. It can be observed that both DF and AF allocate almost all the power to S, when R is in the vicinity of D, and both experience a nearly equal power allocation when R is in the middle. Additionally, $\rho$ increases faster for DF when R moves from the middle towards D than AF in order to achieve the minimum BER at D of the specific system. Furthermore, in case of DF $\rho$ varies for different $E_b/N_0$, whereas it remains almost unchanged for AF. This sensitivity of the power allocation is caused by the application of the non-linear quantization at R.

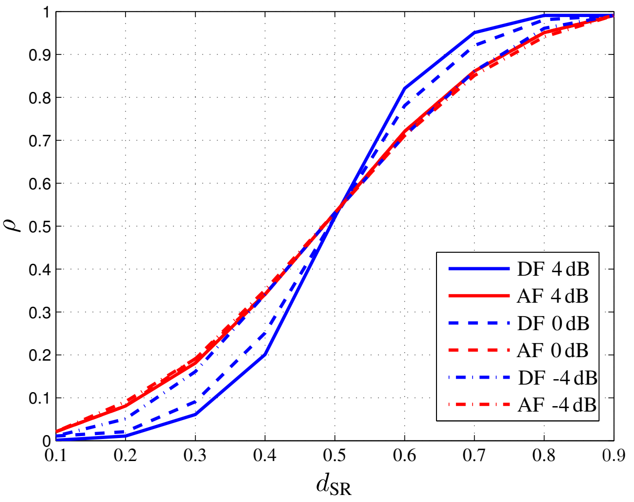


Fig. 4. Power fraction $\rho$ for DF and AF with power allocation for 16-QAM and $E_b/N_0 = -4\,\mathrm{dB}, 0\,\mathrm{dB}, 4\,\mathrm{dB}$, AWGN channel.

Next, the optimized power allocation solutions are applied to link level simulations in order to investigate the achieved BER at D. In Fig. 5, the solid curves represent the BER performances with equally allocated power to S and R ($\rho = 0.5$), while the dashed curves depict those with optimal power allocation for DF. It can be observed, that the BER is efficiently reduced with the help of power allocation for both investigated SNRs. For comparison, analytically achieved BERs are shown indicating the validity of the proposed derivations.

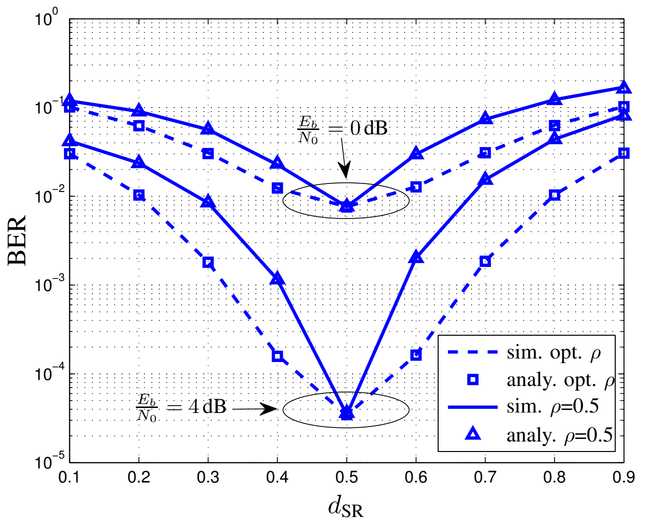


Fig. 5. BER with and without power allocation from simulation and analysis for DF with 16-QAM, AWGN channels, $E_b/N_0 = 0\,\mathrm{dB}, 4\,\mathrm{dB}$.

A comparison of DF and AF with respect to their BER performances at D with and without power allocation is shown in Fig. 6. Obviously, the power allocation solution for AF proposed in [3] leads also to substantial performance improvements. If no power allocation is applied it can be observed, that DF outperforms AF when R is located at distances $d_{\mathrm{SR}} \leq 0.6$ but degrades when R is near to D due to error propagation. By using the proposed power allocation scheme, the performance of DF is improved more significantly compared to AF. Especially, the spatial region in which DF outperforms AF is broadened with optimized power.

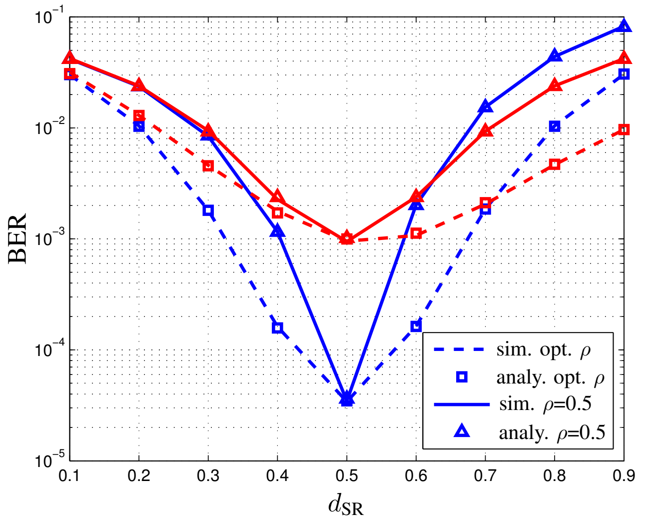


Fig. 6. BER with and without power allocation from simulation and analysis for 16-QAM and $E_b/N_0 = 4\,\mathrm{dB}$, AWGN channels, blue curves for DF and red curves for AF.

Fig. 7 shows for three different relay positions the BER performance for varying SNRs. As expected by Fig. 5 the optimization of power does not improve the performance for $d_{\mathrm{SR}} = 0.5$. However, significant gains can be achieved for the other relaying positions leading to almost the same performance for $d_{\mathrm{SR}} = 0.3$ and $d_{\mathrm{SR}} = 0.7$.

In order to extend the proposed approach for fading channels, instantaneous channel state information is required for the centralized power allocation scheme. Subsequently, Rayleigh fading channels including path-loss are considered. Fig. 8 shows the BERs for DF and AF with and without power allocation for Rayleigh fading as well as for AWGN channels with $d_{\mathrm{SR}} = 0.5$. In contrast to AWGN channels where power allocation leads to no performance gains, the performance is significantly improved by optimizing the power in case of fading channels. As the effective fading gains of the SR and the RD link are usually not the same, the impact of power allocation increases. Additionally, the simulation results show that AF outperforms DF when R is in the middle since fading causes much more severe error propagation at R for DF.

Finally, Fig. 9 shows comparable results for 64-QAM. It can be observed that the BER characteristics for all the considered scenarios remain unchanged in contrast to 16-QAM except for a performance degradation, e.g., 4 dB loss at BER $= 10^{-5}$.

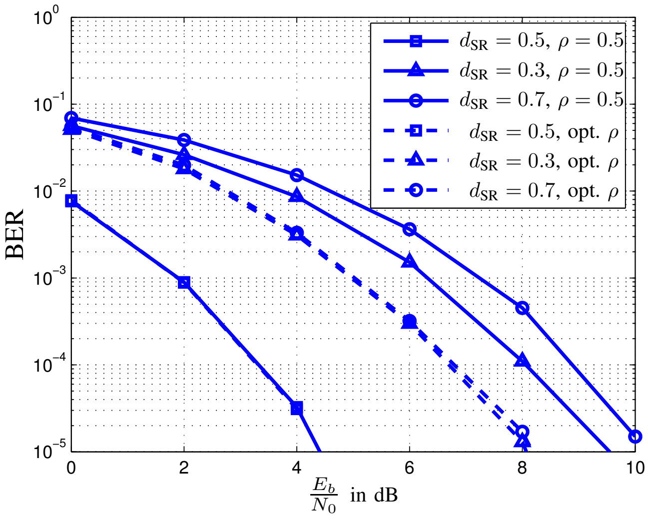


Fig. 7. Simulated BERs with and without power allocation for 16-QAM with DF and varying relay positions $d_{SR} = 0.3, 0.5, 0.7$ for AWGN channels.

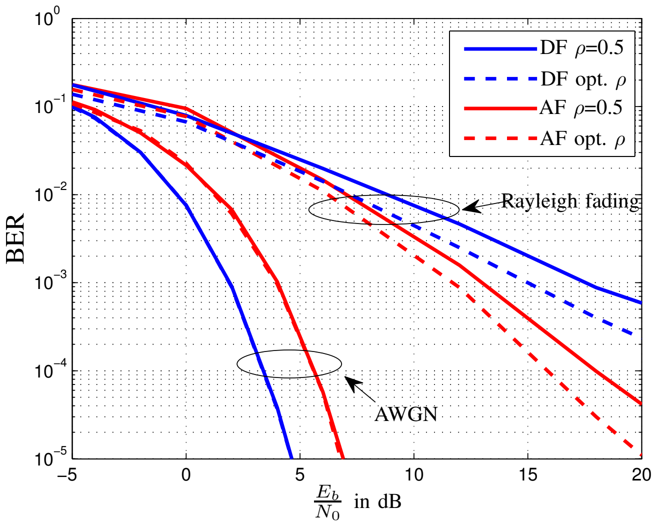


Fig. 8. Simulated BERs with and without power allocation for 16-QAM and $d_{SR} = 0.5$ for AWGN and Rayleigh fading channels, blue curves for DF and red curves for AF.

## VI. Conclusion

In this paper, we first derived a closed-form BER expression for an uncoded single-relay cooperative system using DF and $M$-QAM modulation. For the analysis, the effect of error propagation at the relay on the probability density function of the combined signal at the destination was taken into account. Based on the expression for the BER, the optimized allocation of the total transmit power has been proposed in order to minimize the BER. The improvements of this approach have been demonstrated by simulations.

## VII. Acknowledgement

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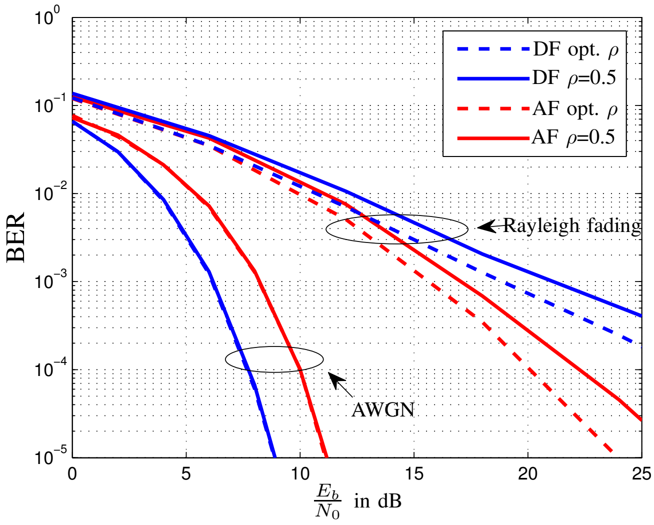


Fig. 9. Simulated BERs with and without power allocation for 64-QAM and $d_{SR} = 0.5$ for AWGN and Rayleigh fading channels, blue curves for DF and red curves for AF.

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