RSCS: Minimum measurement MMV Deterministic Compressed Sensing based on Complex Reed Solomon Coding

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***Abstract*—Compressed Sensing (CS) is an emerging field in communications and mathematics that is used to measure few measurements of long sparse vectors with the ability of lossless reconstruction. In this paper we use methods from channel coding to create the CS recovery algorithm RSCS in the Multiple Measurement Vector case (MMV) that uses a specifically constructed measurement matrix. In particular, we use a modified Reed Solomon encoding-decoding structure to measure sparsely representable vector systems down to the theoretical minimum number of measurements. We prove that the reconstruction is *guaranteed*, even in the low dimensional case.**

***Index Terms*—Compressed Sensing, Reed Solomon, MMV, Deterministic CS**

I. Introduction

A. Motivation

Compressed Sensing (CS) has sparked considerable interest in the last years, especially the Multiple Measurement Vector (MMV) case, that is considering several correlated vectors at once. The main task of CS is to reconstruct long sparsely representable vectors out of few linear measurements, described by a measurement matrix [1]. Many different disciplines like electronics, medicine, physics and communications can make good use of this concept, because it naturally arises in uncountable applications. One of the main challenges in the CS theory is the design of good measurement matrices. Usually random (sub Gaussian) matrices are used, that are well studied and known for good recovery in high dimensions. Sadly in many applications the dimension of the problem is not high enough for pseudo random number generators to work properly or they are not practical to implement. A potential solution is a deterministic choice for the measurement matrix, but an explicit construction that shows good recovery properties is still an open research topic [2], [3], [4]. This work was motivated by the search for an optimal reconstruction algorithm that uses *easily* constructed deterministic matrices and still has good recovery properties in small and large dimensions. Early work of combining channel coding theory with Compressed Sensing [5], [6] inspired us to take the same approach and use this well studied field to tackle the CS framework. Especially Complex Reed Solomon (RS) codes are prominent for their high error correcting capabilities of *sparse* error vectors [7]. This fact can be utilized to solve the CS problem. State of the art algorithms exist that combine Reed Solomon codes with CS, but they only work in the *identity basis*. To improve upon the results in [8] we go one step further by extending this idea to **sparsely representable** vectors that can be sparse in any arbitrary basis.

B. Main Result

Our main result is the derivation of the CS reconstruction algorithm RSCS with a corresponding **deterministic** measurement matrix, that is inspired by Complex Reed Solomon decoding. Let $\mathbf{X} = \mathbf{\Psi C},\ \mathbf{X} \in \mathbb{C}^{N\times L}$ consist of L vectors with N entries each and $\mathbf{C}$ being k-row sparse. Then with no requirements towards $\mathbf{\Psi}$ other than a linear independence of the columns of $\mathbf{\Psi}$ this algorithm is **guaranteed** to retrieve $\mathbf{X}$ with the theoretical minimum of $k+1$ measurements for each of the L vectors if $L \geq k$, even in the low dimensional case. The number of measurements is independent of $\mathbf{\Psi}$ and of the length of $\mathbf{X}$ which makes it increasingly effective for very sparse vectors. Furthermore, there is no need for tuning parameters. The algorithm includes an intrinsic estimator for the usually unknown value k which makes the algorithm easy and efficient to use.

C. Structure

The paper is structured at follows: In section II the CS problem is formally stated and the connection between CS and Reed Solomon codes is explained in the canonical case. In section III this combination is extended towards arbitrary bases. Section IV shows numerical results comparing our contribution to state of the art algorithms. Section V concludes the paper.

II. Theoretical Background

A. Compressed Sensing

Formally spoken, the Multiple Measurement Vector Compressed Sensing (MMV-CS) problem is the reconstruction of $\mathbf{X}$ out of few measurements

$$\mathbf{Y} = \mathbf{\Phi X} \tag{1}$$

for the dense vector system $\mathbf{X} \in \mathbb{C}^{N\times L}$ being *row sparsely representable*. Formally this means

$$\exists \mathbf{\Psi} \in \mathbb{C}^{N\times N},\ \text{s.t. } \mathbf{X} = \mathbf{\Psi C} \text{ with } ||\mathbf{C}_l||_0 = k \ll N \tag{2}$$

for all columns of $\mathbf{C}$ ($l \in 1, \cdots, L$) and

$$\text{supp } \mathbf{C}_i = \text{supp } \mathbf{C}_j \ \forall i, j \in 1, \cdots, N. \tag{3}$$

The measurement matrix $\mathbf{\Phi}$ is an $m \times N$-matrix with $m < N$ and consequently $\mathbf{Y}$ lies in $\mathbb{C}^{m \times L}$. We define

$$\mathcal{I} := \{i \in \mathbb{N} | C_i \neq 0 \ \forall i \in 1, \cdots, N\} \tag{4}$$

as the k dimensional support of $\mathbf{C}$ and $\mathbf{C}_i$ as the corresponding i-th **row** of $\mathbf{C}_\mathcal{I}$, formally $\mathbf{C}_i := \mathbf{C}_{\mathcal{I}(i)}$.
To further differentiate between the k dimensional support and the N dimensional vector space all other vectors will still use the $\mathcal{I}(i)$ subset notation.
The equation system (1) can not be solved uniquely, instead the *sparsest* solution $\mathbf{C}$ should be used. The joint properties of $\mathbf{A} := \mathbf{\Phi\Psi}$ then determine the recovery properties. For this a firm mathematical theory towards $\mathbf{A}$ is needed that can guarantee perfect recovery of every sparsely representable vector system $\mathbf{X}$ when specific conditions are fulfilled. To classify if a given matrix is well suited Candes [9] introduced the now ubiquitous **Null Space Property**, that can be reformulated for the MMV case [10]:

Theorem 1 (Null Space Property)**.** *Let $\mathbf{A} \in \mathbb{R}^{m \times N}$ be a measurement matrix and $\mathbf{Y} \in \mathbb{R}^{m \times L}$ be the corresponding L measurements. If*

$$\textit{spark}(\mathbf{A}) \geq k + \lceil k/L \rceil \tag{5}$$

*(with $\lceil k \rceil$ as rounding up k to the next integer) there is an unique k-row sparse $\mathbf{X} \in \mathbb{R}^{N \times L}$, s.t. $\mathbf{Y} = \mathbf{AX}$. The spark is the **minimum** number of linear **dependent** columns. Formally that means*

$$\textit{spark}(\mathbf{A}) = \min_{\mathbf{x} \neq 0} ||\mathbf{x}||_0 \ \textit{s.t.} \ \mathbf{Ax} = 0 \tag{6}$$

Due to $\textit{spark}(A) \leq m$, it follows that

$$m \geq k + \lceil k/L \rceil \tag{7}$$

needs to be fulfilled for unique recovery. For $L \geq k$, this means that

$$m \geq k + 1 \tag{8}$$

is needed. A proof can be found in [10].

Randomly chosen matrices like subgaussian ones have full spark (spark$(\mathbf{A}) = m$) with high probability, so many researchers use these matrices as sensing matrix $\mathbf{\Phi}$. This way we can make the assumption towards $\mathbf{\Psi}$ that $\mathbf{A} := \mathbf{\Phi\Psi}$ is still subgaussian, so that the reconstruction task shifts from the complicated dense $\mathbf{X}$ to the sparse $\mathbf{C}$. This way, only the easier task of reconstructing the **sparse** vector $\mathbf{C}$ remains.
Sadly this does not allow for an simple implementation because the reconstruction algorithm needs to be aware of the random matrix that was used to measure. To obtain an efficient hardware friendly algorithm in this paper we concentrate on finding **deterministic** measurement matrices. The theory of Complex Reed Solomon Codes allows us to define such a matrix $\mathbf{\Phi}$ but for this to work a careful design of the algorithm needs to be considered. This way the deterministic matrix can be hard coded in the encoder and the decoder, leading to a simple and efficient design. The next subsection will introduce Complex Reed Solomon (RS) coding and then combine CS with the RS theory.

B. Interleaved Complex Reed Solomon Coding

To clarify the main idea, in this section $\mathbf{\Psi}$ is assumed to be the identity matrix $\mathbf{I}$ and the number of signals L is set to be equal to the sparsity k. In other words the vector system $\mathbf{X} = \mathbf{C}$ is directly sparse in the canonical basis. Later on we will generalize these conditions.
Complex Reed Solomon (RS-) codes belong to the family of cyclic block codes [11]. If only a few of the complex symbols are erroneous, they can be reconstructed, regardless of the actual complex number in each incorrect symbol. To combine CS with CRS codes, we interpret the systematic part of the interleaved code as the sparse vector system $\mathbf{X}$, while the measurements $\mathbf{Y}$ are interpreted as the parity symbols. This way, each complex number in our original sparse vectors $\mathbf{X}$ is one RS-symbol. Interleaved Complex RS codes are prominent for correcting Lk symbol errors for $L \geq k$ when $k + 1$ parity symbols are given for each individual vector, if the positions and values of the erroneous symbols are unknown. Figure 1 illustrates the connection of RS decoding and CS reconstruction for $L = k$.

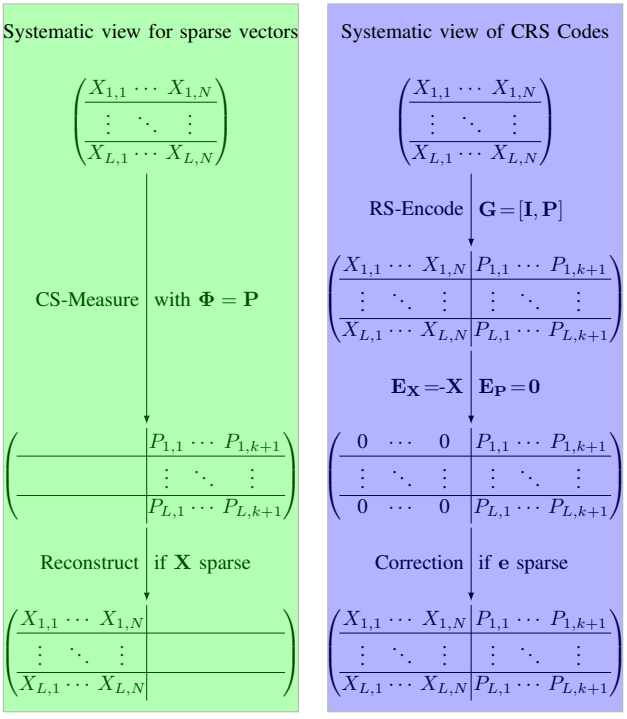

Fig. 1. Schematic for measurement and reconstruction of sparse vectors for $\mathbf{\Psi} = \mathbf{I}$

The left part shows a CS interpretation of the problem. A sparse vector system $\mathbf{X}$ is measured by a matrix $\mathbf{\Phi}$ to obtain $L(k + \lceil L/k \rceil) = k(k+1)$ measurements $\mathbf{P}$ which are sufficient to reconstruct $\mathbf{X}$ even with noise.

The right part describes the RS view. RS decoders with $L(k+\lceil L/k \rceil) = k(k+1)$ parity symbols can correct up to Lk errors at any k positions and amplitudes, so especially the total erasure of the systematic part $[\mathbf{E_X}, \mathbf{E_P}] = [-\mathbf{X}, \mathbf{0}]$ is a correctable error, if $\mathbf{X}$ is **row sparse**. In other words, for $L \geq k$ the CS algorithm can measure $k+1$ parity symbols of each of the L systematic Reed Solomon encoding schemes and **guarantee** the recovery, if $\mathbf{C}$ is k-row sparse. The number of parity symbols is independent of the length of the original vector system $\mathbf{X}$, so its well suited for long vectors with $k \ll N$. To better formally introduce the RS coding idea, we define $\alpha = \exp(2\pi j/N)$ as the generating element of order N, $\mathbf{G}$ as the generator matrix of the RS code and the encoding as

$$[\mathbf{X}_l, \mathbf{P}_l] = \mathbf{G}\mathbf{X}_l \tag{9}$$

for each corresponding RS code word ($l = 1, \cdots, L$) with parity symbols $\mathbf{P}$. With $[\mathbf{f}](t)$ denoting the evaluation of the polynomial with coefficients $[\mathbf{f}]$ at t, the parity symbols need to fulfill the equation

$$[\mathbf{X}_l, \mathbf{P}_l](\alpha^j) = 0 \quad \forall j = 1, \ldots, k+1, l = 1, \ldots, L. \tag{10}$$

Further defining $\mathbf{E}_l$ as the additive error and $\mathbf{R}_l = [\mathbf{X}_l, \mathbf{P}_l] + [\mathbf{E}_{l,\mathbf{X}}, \mathbf{E}_{l,\mathbf{P}}]$ as the received vector, we can use the standard RS syndrome decoding idea, i.e.

$$S_j = \mathbf{R_l}(\alpha^j) = \underbrace{[\mathbf{X}_l, \mathbf{P}_l](\alpha^j)}_{=0} + [\mathbf{E}_{l,\mathbf{X}}, \mathbf{E}_{l,\mathbf{P}}](\alpha^j) \tag{11}$$

for $j = 1, \ldots, k+1, l = 1, \ldots, L$ to reconstruct each error vector. In terms of CS only the parity part is known, meaning that the systematic part can be seen as totally erased by an error vector $[\mathbf{E_X}, \mathbf{E_P}] = [-\mathbf{X}, \mathbf{0}]$. Thus, the decoder simply has to find $\mathbf{E}$ to reconstruct the total system $\mathbf{X}$. Consequently, the decoding is actually only based on this erasure hypothesis, so all syndromes can be directly computed by one CS measurement step. Formally, this results in

$$\begin{pmatrix} S_0 \\ \vdots \\ S_k \end{pmatrix} = \begin{pmatrix} -\mathbf{X}(\alpha^0) \\ \vdots \\ -\mathbf{X}(\alpha^k) \end{pmatrix} = - \begin{pmatrix} \alpha^0 & \cdots & \alpha^{0N} \\ \vdots & \ddots & \vdots \\ \alpha^k & \cdots & \alpha^{kN} \end{pmatrix} \mathbf{X}. \tag{12}$$

In this way, the complete encoding step is cast in the same way as in equation (1), the standard CS problem. In other words, we can interpret the computation of syndromes as a CS compatible sensing matrix $\mathbf{\Phi}$ with N columns and $k+1 \ll N$ rows. Namely $\mathbf{\Phi}$ is the Fourier matrix

$$\mathbf{\Phi} = \begin{pmatrix} \alpha^0 & \cdots & \alpha^{0N} \\ \vdots & \ddots & \vdots \\ \alpha^k & \cdots & \alpha^{kN} \end{pmatrix} \tag{13}$$

with known full spark [12], thus fulfilling theorem 1. Because of the equivalence of both views, this sensing matrix still allows us to use known algorithms like the Berlekamp Massey algorithm [7] for the reconstruction of the original sparse vector, effectively combining the two theories. The number of measurements $m = k+1$ is the theoretical minimum for guaranteed recovery and independent of the length of the original vector system $\mathbf{X}$.

To really combine CS with RS, one crucial detail needs to be clarified. The previous derivations all assumed $\mathbf{X}$ to be sparse, but CS generally includes the vast group of sparsely representable vectors of the form

$$\mathbf{X} = \mathbf{\Psi}\mathbf{C},\ \mathbf{C} \text{ sparse}. \tag{14}$$

In general, there are not exactly k nonzero elements in $\mathbf{X}$ but k columns of $\mathbf{\Psi}$ are active, e.g. k frequencies. For the algorithm to still work in this more general case, it needs to be extended as presented in the next section.

III. Combination of Interleaved Complex Reed Solomon Codes and MMV CS

A. Recap of RS decoding

To explain the extension of the reconstruction algorithm in [8] we again start with the canonical basis. That is, at first each vector $\mathbf{X}_l$ has the form

$$\mathbf{X}_l = \sum_{i=1}^{k} C_{i,l} \mathbf{I}_{\mathcal{I}(i)} \tag{15}$$

with $\mathbf{I}_{\mathcal{I}(i)}$ being the columns of the identity matrix. In other words, each column of $\mathbf{X}$ is sparse in the canonical basis. The first main step of the standard Complex Reed Solomon decoder is the calculation of the error locator polynomial (ELP) of the form

$$\Lambda(t) = \prod_{i=1}^{k} (t - \alpha^{\mathcal{I}(i)}) = t^k + \Lambda_1 t^{k-1} + \Lambda_2 t^{k-2} + \cdots + \Lambda_k \tag{16}$$

with $t = \alpha^{\mathcal{I}(i)}$ as roots for the active set $\mathcal{I}$ of nonzero entries. Inserting the roots leads to k equations for the parameters $\Lambda_1, \ldots, \Lambda_k$:

$$\begin{pmatrix} \alpha^{\mathcal{I}(1)\cdot(k-1)} & \cdots & \alpha^{\mathcal{I}(1)\cdot 0} \\ \vdots & \ddots & \vdots \\ \alpha^{\mathcal{I}(k)\cdot(k-1)} & \cdots & \alpha^{\mathcal{I}(k)\cdot 0} \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} -\alpha^{\mathcal{I}(1)\cdot k} \\ \vdots \\ -\alpha^{\mathcal{I}(k)\cdot k} \end{pmatrix} \tag{17}$$

The syndromes (12) that can be used to evaluate these equations are of the form

$$S_{j,l} = \mathbf{\Phi}_j \mathbf{X}_l = \begin{pmatrix} \alpha^j & \cdots & \alpha^{jN} \end{pmatrix} \left(\sum_{i=1}^{k} C_{i,l} \mathbf{I}_{\mathcal{I}(i)} \right) = \sum_{i=1}^{k} C_{i,l} \alpha^{\mathcal{I}(i)\cdot j} \tag{18}$$

so multiplying by $\mathbf{C}_i$ and different powers of α before adding the rows in (17) leads to

$$\begin{pmatrix} S_{k-1} & \cdots & S_0 \\ \vdots & \ddots & \vdots \\ S_{2k-2} & \cdots & S_{k-1} \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} -S_k \\ \vdots \\ -S_{2k-1} \end{pmatrix} \tag{19}$$

The rank of this system is the number of nonzero elements of $\mathbf{X}$. If this number is less than the maximum $k = m/2$, the matrix has to be transformed appropriately to get a solvable full rank system. With all Λ_i being computed the generalized ELP can be computed. The roots of the ELP now describe the active set $\mathcal{I}$. With $\mathcal{I}$ known, the corresponding $\mathbf{C}_i$ can be easily derived by L_2-minimization.

B. Modification to sparsely representable vectors

Now we extend this algorithm to $\mathbf{X}_l$ of the form

$$\mathbf{X}_l = \sum_{i=1}^{k} C_{i,l} \mathbf{\Psi}_{\mathcal{I}(i)} \tag{20}$$

for arbitrary $\mathbf{\Psi}$. We propose a modification of the error locator polynomial to take this change in account. With the definition $\mathbf{A}_\alpha = \mathrm{diag}(\alpha, \alpha^2, \ldots, \alpha^N)$ we propose the generalization

$$\Lambda_{\mathbf{\Psi}}(\mathbf{t}) = \mathbf{1}\left(1 \cdot \mathbf{A}_\alpha^k + \Lambda_1 \mathbf{A}_\alpha^{k-1} + \ldots + \Lambda_k\right)\mathbf{t} \tag{21}$$

with $\mathbf{t} = \mathbf{\Psi}_{\mathcal{I}(i)}$ as root for the active set $\mathcal{I}$. Because $\mathbf{t} \in \mathbb{C}^{N\times 1}$ it follows that

$$\Lambda_{\mathbf{\Psi}}(\mathbf{t}) : \mathbb{C}^{N\times 1} \to \mathbb{C}^1 \tag{22}$$

With the equation

$$\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j) = \begin{pmatrix}\alpha^{1\cdot j} & \cdots & \alpha^{N\cdot j}\end{pmatrix} \mathbf{\Psi}_{\mathcal{I}(i)} \tag{23}$$

we can interpret the condition for the roots as a linear equation as

$$\begin{pmatrix} \mathbf{\Psi}_{\mathcal{I}(1)}(\alpha^{k-1}) & \cdots & \mathbf{\Psi}_{\mathcal{I}(1)}(\alpha^0) \\ \vdots & \ddots & \vdots \\ \mathbf{\Psi}_{\mathcal{I}(k)}(\alpha^{k-1}) & \cdots & \mathbf{\Psi}_{\mathcal{I}(k)}(\alpha^0) \end{pmatrix} \begin{pmatrix}\Lambda_1 \\ \vdots \\ \Lambda_k\end{pmatrix} = \begin{pmatrix} -\mathbf{\Psi}_{\mathcal{I}(1)}(\alpha^k) \\ \vdots \\ -\mathbf{\Psi}_{\mathcal{I}(k)}(\alpha^k) \end{pmatrix} \tag{24}$$

This extension is consistent, because if $\mathbf{\Psi}$ is the identity matrix, the modified equation system and the corresponding error locator polynomial lead to the exact same result as the original version. Namely

$$\Lambda_{\mathbf{I}}(\mathbf{I}_i) = \Lambda(\alpha^i) \tag{25}$$

But if $\mathbf{\Psi}$ is not equal to $\mathbf{I}$ as is the case in the general setting, this modification can still be used to utilize the syndromes in the MMV case. Multiplying the equations by $\mathbf{C}_i$ before adding them leads to the corresponding syndrome equation

$$S_{j,l} = \mathbf{\Phi}_j \mathbf{X}_l = \left(\alpha^j \cdots \alpha^{jN}\right)\left(\sum_{i=1}^{k} C_{i,l}\mathbf{\Psi}_{\mathcal{I}(i)}\right) \tag{26}$$

$$= \sum_{i=1}^{k} C_{i,l}\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j) \tag{27}$$

as in equation (18), so

$$\begin{pmatrix} S_{k-1} & \cdots & S_0 \end{pmatrix}\begin{pmatrix}\Lambda_1 \\ \vdots \\ \Lambda_k\end{pmatrix} = \begin{pmatrix} S_k \end{pmatrix} \tag{28}$$

emerges. Sadly this does not work for higher syndromes, because the condition

$$\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j) = \mathbf{\Psi}_{\mathcal{I}(i)}(\alpha)^j, \quad j = 1, \ldots, m \tag{29}$$

is not fulfilled for arbitrary $\mathbf{\Psi}$. But due to the MMV case every vector $\mathbf{X}_j$ has the same support, we can use $k+1$ syndromes from each vector to build the error locator polynomial as long as $L \geq k$. In other words, k vectors with $k+1$ measurements each are sufficient for the perfect reconstruction of the original vector system by solving the equation system for all Λ_i and testing the columns of $\mathbf{\Psi}$ in the modified ELP. This again also effectively solves the problem of finding the unknown sparsity parameter k. Analog to the normal RS decoder, the rank of the linear equation system of syndromes determines the sparsity, so **the sparsity does not need to be known at the decoder entrance**.

The Fourier matrix has nice properties, that can be exploited as designed above. But in reality, other designs like binary or Rademacher matrices are preferred due to hardware restrictions or easier theoretical arguments. The next section will depict how the same Reed Solomon approach can be used for arbitrary full spark sensing matrices.

C. Modification to arbitrary sensing matrices

$\mathbf{Y} = \mathbf{\Phi}\mathbf{\Psi}\mathbf{C}$ can be reconstructed for the Fourier matrix $\mathbf{\Phi}$ and *arbitrary* $\mathbf{\Psi}$. To change the Fourier matrix to any other matrix, the change needs to be transferred to $\mathbf{\Psi}$, s.t.

$$\underline{\mathbf{\Phi}}\,\underline{\mathbf{\Psi}} = \mathbf{\Phi}\mathbf{\Psi} \tag{30}$$

This way, the only step in the algorithm that needs to be changed is the evaluation of the generalized ELP with some new $\underline{\mathbf{\Psi}}$ instead of $\mathbf{\Psi}$.

More formally, this means: If the Fourier matrix $\mathbf{\Phi}$ is changed to $\underline{\mathbf{\Phi}}$, then

$$\mathbf{\Phi}\mathbf{\Psi} = \mathbf{\Phi}\underbrace{\mathbf{A}\mathbf{A}^{-1}}_{I}\mathbf{\Psi} = \underbrace{\mathbf{\Phi}\mathbf{A}}_{\underline{\mathbf{\Phi}}}\underbrace{\mathbf{A}^{-1}\mathbf{\Psi}}_{\underline{\mathbf{\Psi}}} \tag{31}$$

The choice

$$\mathbf{A} = \mathrm{pinv}(\mathbf{\Phi})\underline{\mathbf{\Phi}} + \mathrm{null}(\mathbf{\Phi})\mathrm{null}(\mathbf{\Phi})^\top \tag{32}$$

solves the equation $\underline{\mathbf{\Phi}} = \mathbf{A}\mathbf{\Phi}$, so the corresponding modified dictionary is $\underline{\mathbf{\Psi}} = \mathbf{A}^{-1}\mathbf{\Psi}$.

The next section will now compare our contribution with other state of the art algorithms.

IV. Comparison to Other CS algorithms

To evaluate our new proposition, we compare it to existing MMV algorithms, namely SOMP [13] and MUSIC [14]. Figure 2 shows the phase diagram for $N = 100$ and $L = 25$ for the range $k = 1, \ldots, 50$ and $m = 1, \ldots, 50$ in the noiseless case. The plot shows for each m/N the highest k/m value that lead to an overall recovery success rate of over 50 percent, averaged over 1000 Monte Carlo trials. This standard measure is independent of N so the phase diagram is a valuable comparison tool. As theorem 1 shows, the optimal line reachable is $k/m = k/(k+1)$ as depicted. As one can see, the optimal line is reached by both our RSCS approach and the MUSIC algorithm as long as $k \leq L$. Consistent to the theory this is the range of guaranteed recovery. After this point, as shown by the red dashed line, no guarantee can be given so the phase diagram shows a strong decline. The SOMP algorithm on the other hand does not reach the optimal line at all, always falling short in comparison to the other competitors.

To better compare our contribution in a more realistic case, figure 3 shows results for the three different algorithms in the noisy case. The simulation ran for $N = 100$, $k = 10$,

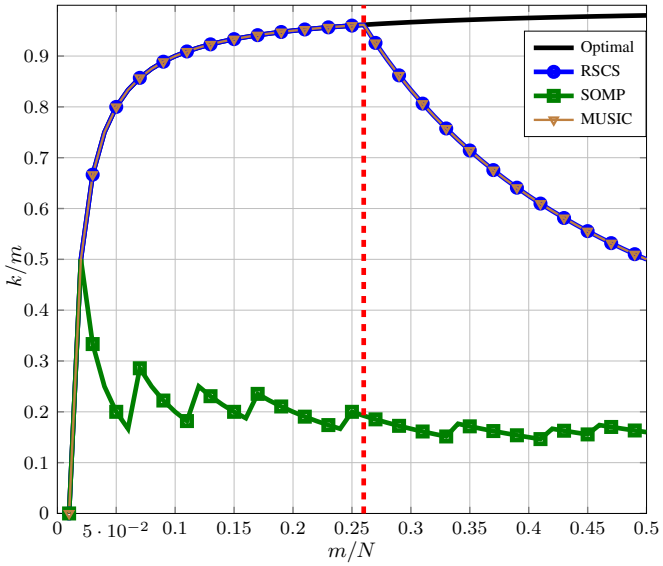

Fig. 2. Phase diagram, compared for RSCS (our contribution), SOMP and MUSIC in the setting $N = 100$, $L = 25$ for $k = 1, \ldots, 50$ and $m = 1, \ldots, 50$

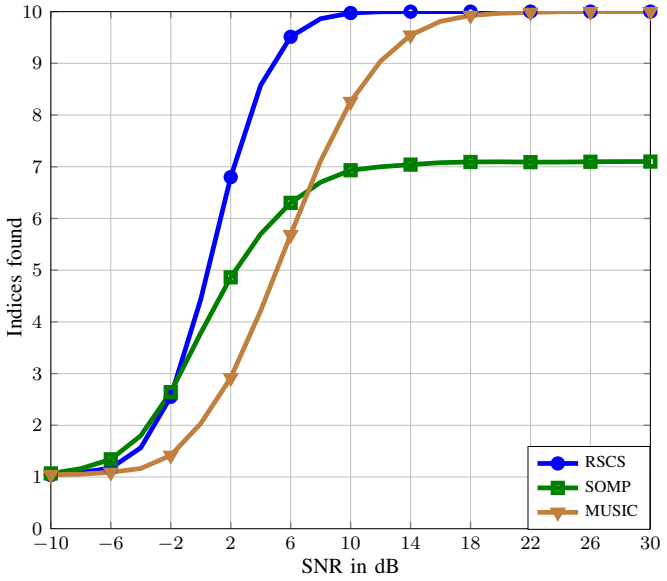

Fig. 3. Mean number of the reconstructed support indices, compared for RSCS (our contribution), SOMP and MUSIC in the setting $N = 100$, $k = 10$, $m = 20$, $L = 10$ for the signal to noise ratio from -10 to 30.

$m = 20$ and $L = 10$. As a measure of performance we depict the mean number of correctly estimated support entries averaged over 1000 Monte Carlo trials. As one can see, our Reed Solomon approach (RSCS) performs best. If the SNR is above 10dB, the reconstruction is error free every time. The MUSIC algorithm also achieves perfect reconstruction, but needs a minimal SNR of 20 dB to guarantee it. The SOMP algorithm cannot guarantee reconstruction. Even in the noiseless case, only 70% of the indices can be reconstructed in the mean. So even in this mild setting and with optimized tuning parameters, SOMP fails.

V. Conclusion

We showed that a generalization of the Reed Solomon decoding algorithm can be used as a powerful CS algorithm in the MMV case. It only needs the theoretical minimum number of $k + 1$ measurements for guaranteed reconstruction as long as the amount of simultaneous vectors L is higher than k. The algorithm even works for sparsely representable vectors that are sparse in any arbitrary basis. Future work will try to lower the demand towards $L \geq k$ even further to transform it into an universal CS algorithm.

VI. Acknowledgment

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