Channel Coding II

Exercises

– WiSe 2019/2020 –

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1 Concatenated Codes

Exercise 1.1 **Decoding sequence of linear product codes**

Considering concatenated codes as in chapter 1, the succession of the encoding and the decoding of linear constituent codes is reversible. This has the effect that all columns and all rows represent valid code words of the respective code. Show this for linear block codes with the help of matrix calculus and check your results by means of the example given in the lecture slides (concatenation of $(3, 2, 2)$- and $(4, 3, 2)$-SPC codes). Note that the block interleaver has to be taken into consideration.

Exercise 1.2 ***L*-algebra**

a) The definition of *log-likelihood*-ratio (LLR) is assumed to be known (see lecture slides). Prove

$$L\left(\sum_{i=1}^{n} \oplus u_i\right) = \ln \frac{\prod_{i=1}^{n}(e^{L(u_i)}+1) + \prod_{i=1}^{n}(e^{L(u_i)}-1)}{\prod_{i=1}^{n}(e^{L(u_i)}+1) - \prod_{i=1}^{n}(e^{L(u_i)}-1)} \tag{1}$$

by induction.

b) Calculate the expected value $\mathrm{E}\{\hat{x}\}$ for the estimation of a BPSK symbol x given the corresponding LLR $L(\hat{x})$. Plot the value $\mathrm{E}\{\hat{x}\}$ for different signal-to-noise ratios by assuming the range $-2 : 0.1 : 2$ for the receive symbols y and the range $-2 : 2 : 10$ dB for E_s/N_0.
Hint: For equiprobable input symbols the LLR $L(\hat{x})$ is determined by the value $L(y|x) = 4\alpha \frac{E_s}{N_0} y$ and we assume $\alpha = 1$ in this exercise.

Exercise 1.3 **Comparison of the exact solution and approximation of LLR combining**

Create a MATLAB function [exact, approx] = `llr`(L) which determines the LLRs of the combination of several statistically independent symbols exactly (2) and approximately (3).

$$L(u_1 \oplus \ldots \oplus u_n) = 2\,\mathrm{artanh}\left(\prod_{i=1}^{n} \tanh(L(x_i)/2)\right) \tag{2}$$

$$\approx \min_j(|L(x_j)|) \cdot \prod_{i=1}^{n} \mathrm{sgn}\,(L(x_i)) \tag{3}$$

Compare the exact solution and the approximate one by assuming the symbols $\mathbf{y}_1 = -2 : 0.1 : 2$ and $\mathbf{y}_2 = 0.2 : 0.2 : 1$. Determine the LLRs and plot the result against $\mathbf{y}_1$ for $E_s/N_0 = 2$ dB.

Exercise 1.4 **Soft-output decoding of SPC codes**

a) The information word $\mathbf{u} = (1,\ 0,\ 1)$ is encoded with a (4,3,2)-SPC code, BPSK-modulated and subsequently transmitted over an AWGN channel with a signal-to-noise ratio of $E_s/N_0 = 2$ dB. At the receiver the sequence $\mathbf{y} = (-0.8,\ 1.1,\ 0.3,\ 0.4)$ is observed. Determine the LLRs with the routine `llr.m` from exercise 1.3 and decode the receive vector. What is the result?

b) Now decode with the approximation solution. Compare the result with a).

c) Determine the probabilities for a correct decoding decision.

Exercise 1.5 **BCJR decoding of a convolutional code**

Assume that $\mathbf{y} = [-0.6727, -0.8254, 0.8133, -0.2742, 0.4117, 1.1832, -1.1364, -0.8861]$ has been received after transmission over an AWGN channel with a noise variance of $\sigma_n^2 = 1$. On the transmitter side the binary information word $\mathbf{u} = [1, 1, 0, 0]$ has been encoded with a $[5, 7]_8$ convolutional code (NSC), BPSK-modulated and finally transmitted. Perform BCJR decoding under the assumption that the last state is known to be $(0, 0)$. For ease of calculation use the Max-Log-MAP in the logarithmic domain. (Hint: Draw a full trellis first, then follow the pertinent slides of Chapter 1, CC II.)

Exercise 1.6 **Decoding of a modified product code**

Given is a modified product code consisting of two $(3, 2, 2)$-SPC codes. The following 2×2 information matrix describes the four information bits

$$\mathbf{U} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the BPSK-modulated code bits are shown in the code matrix

$$\mathbf{X} = \begin{pmatrix} +1 & +1 & +1 \\ +1 & -1 & -1 \\ +1 & -1 & \end{pmatrix}.$$

After transmission over an AWGN channel with $E_s/N_0 = 2$ dB, we get the following receive matrix

$$\mathbf{Y} = \begin{pmatrix} -1.5 & +1.5 & +1.2 \\ +1.1 & +1.0 & -1.5 \\ +0.5 & -2.5 & \end{pmatrix}.$$

Decode this product code step by step using MATLAB.

Exercise 1.7 **Decoding of a (7,4,3)-Hamming code using an LDPC decoder**

Given is a (7,4,3)-Hamming code with the generator matrix

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and the corresponding parity check matrix

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

a) Linear block codes can be represented by a Factor Graph. It is a bipartite graph with

- a *variable node* for each code symbol,
- a *check node* for each check equation,
- an *edge* between a variable node and a check node if the corresponding symbol participates in the pertinent check equation.

A *cycle* is a closed path through the graph that begins and ends at the same variable node. The length of a cycle is the number of edges traversed.

- Give the corresponding Factor Graph for the Hamming code with the parity check matrix $\mathbf{H}$.
- How large is the minimum length of a cycle (the *girth* of the Factor Graph)?

b) Determine the sets $\mathcal{K}_k$ and $\mathcal{J}_j$ which provide the connection between the variable nodes and the check nodes by $\mathcal{K}_k = \{j : H_{k,j} = 1\}$ and $\mathcal{J}_j = \{k : H_{k,j} = 1\}$, where $k = 0, 1, \ldots, K-1$ and $j = 0, 1, \ldots, J-1$ with K and J being the number of check nodes and the variable nodes, respectively.

An information sequence $\mathbf{u}$ = [1, 1, 0, 1] is encoded with $\mathbf{c} = \mathbf{u} \cdot \mathbf{G}$. The code word $\mathbf{c}$ is BPSK-modulated via $\mathbf{x} = 1 - 2 \cdot \mathbf{c}$. The sequence $\mathbf{x}$ is then transmitted over an AWGN channel with the signal-to-noise ratio of $E_s/N_0 = 2$ dB. Assume that the receive signal $\mathbf{y} = \mathbf{x} + \mathbf{n}$ is given by $\mathbf{y}$ = [-1.3, -0.4, 1.1, -1.2, 0.6, 0.3, 0.7].

c) Calculate and collect the extrinsic information (of the first iteration) at each check node from the connected variable nodes. Use the boxplus approximation.

$$E_j^k = \sum_{i \neq j,\, i \in \mathcal{K}_k}^{\boxplus} L(\hat{x}_i)$$

d) Collect the extrinsic information at each variable node from the connected check nodes.

$$a_j = \sum_{k \in \mathcal{J}_j} E_j^k$$

e) Make a decision at each variable node.

f) Implement in MATLAB a simulation chain wherein the decoding is done iteratively and compare your results with the hard decoding scheme for the Hamming code.