Energy Efficiency: Rate Splitting vs. Point-to-Point Codes in Gaussian Interference Channels

(Invited Paper)

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***Abstract*—Communication in networks is subject to the fundamental limits of reliable communication. This limits not only the maximum transmission rate but also the energy efficiency (EE). The goal of this paper is to assess the absolute limits of energy-efficient communication in Gaussian interference channels. In all interference regimes with known sum-capacity, this limit is achievable by point-to-point (PTP) codes. However, in contrast to the sum-capacity, we argue that rate splitting is strictly less energy-efficient than PTP codes in these regimes due to its higher decoding complexity. The only interference regime with unknown sum-capacity is the case with moderate interference. We show numerically that PTP codes are not always sufficient to maximize the EE since rate splitting offers better EE in some moderate interference scenarios. Computing these EEs requires the globally optimal solution of several challenging non-convex optimization problems. For this purpose, we apply the novel mixed monotonic programming framework that allows a more efficient solution of these problems than state-of-the-art approaches like monotonic fractional programming.**

***Index Terms*—Resource allocation, energy efficiency, global optimization, interference channel, rate splitting**

I. Introduction

Energy efficiency (EE) is a main objective in the design of modern communication networks [1]. A key building block of these networks are Gaussian interference channels (GICs) [2]. Their analysis provides deep insights into (wireless) network design while still being reasonably tractable. Despite four decades of research, the complete characterization of its capacity region, i.e., the fundamental limits of reliable communication, remains an open problem. Focusing only on the sum-rate (or throughput), the situation is much more satisfying with sum-capacity results being available for most interference regimes. This sum-capacity plays a prominent role in maximizing the global energy efficiency (GEE), the goal of this paper.

Recently, rate splitting has been shown to increase spectral and energy efficiency in practical wireless networks [3], [4]. The concept is much older though, being one main ingredient of the famous Han-Kobayashi (HK) coding scheme [5] for the interference channel. Its advantage over simpler point-to-point (PTP) codes is that decoders are not forced to decide between completely treating an interfering message as noise or decoding it. Instead, HK coding, and especially the rate splitting part, softly bridges the gap between both interference mitigation approaches by allowing to decode interfering messages partially and treat the remaining part as noise. It leads to the largest known achievable rate region for the interference channel and is within one bit per user of the GIC's capacity region [6]. It also achieves the sum-capacity in all cases with established sum-capacity. However, PTP codes are also sum-capacity achieving in these cases, and, due to their lower decoding complexity, decoding them requires less energy than HK codes while achieving the same throughput. Thus, HK coding can not achieve the same GEE as PTP codes if the employed PTP code achieves the sum-capacity. In the moderate interference regime though, the only interference regime without known sum-capacity and an important scenario for multi-cell wireless networks, HK coding might provide a benefit over PTP codes.

We are interested in maximizing the GEE of the GIC. This requires the globally optimal solution of several challenging non-convex optimization problems. For this purpose, we introduce a novel approach that allows a more efficient solution of these problems than state-of-the-art approaches like monotonic fractional programming [7]. In the next section, we formally introduce the system model and review relevant capacity results. Then, we discuss GEE maximization in the GIC and explore the connection of the GEE to the sum-capacity. In Section IV, we introduce the *mixed monotonic programming* framework and apply it to the optimization problems posed in Section III. We conclude this paper by presenting numerical results that demonstrate EE gains due to HK coding in Section V.

II. System Model and Preliminaries

We consider a complex GIC in standard form [2]

$$y_1 = x_1 + a_2 x_2 + z_1, \qquad y_2 = a_1 x_1 + x_2 + z_2, \qquad (1)$$

where, for $i = 1, 2$, a_i is the complex-valued channel crosstalk coefficient, z_i is independent and identically distributed (i.i.d.) zero-mean circularly symmetric complex Gaussian noise with

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unit power, and the transmitted signal x_i is subject to an average power constraint P_i.

The capacity region of this channel when constrained to capacity-achieving PTP codes[1] is the union of the achievable rate regions with proper Gaussian codebooks and where each receiver either uses joint decoding (JD) or treating interference as noise (TIN) [8, Thm. 2]. Define

$$\mathcal{R}_{1,\mathrm{TIN}}(\boldsymbol{p}) = \left\{\boldsymbol{R} : 0 \leq R_1 \leq \mathrm{C}\left(\frac{P_1}{1+|a_2|^2 P_2}\right)\right\} \tag{2}$$

and

$$\mathcal{R}_{1,\mathrm{JD}}(\boldsymbol{p}) = \left\{\boldsymbol{R} : \begin{array}{l} 0 \leq R_1 \leq \mathrm{C}(P_1), \\ R_1 + R_2 \leq \mathrm{C}\big(P_1 + |a_2|^2 P_2\big) \end{array}\right\}, \tag{3}$$

where $\mathrm{C}(x) = \log_2(1+x)$. Then, the PTP capacity region of the GIC is

$$\mathcal{C}_{\mathrm{ptp}} = \bigcup_{\boldsymbol{d}\in\{\mathrm{TIN},\mathrm{JD}\}^2} \bigcap_{i=1}^{2} \mathcal{R}_{i,d_i}(\boldsymbol{P}). \tag{4}$$

This is also the capacity region under the constraint of random codebook ensembles and coded time sharing [9].

Instead, the largest rate region achievable when constrained to random code ensembles, rate splitting, superposition coding, and coded time sharing is the HK region [9]. Each transmitter splits its message into common and private parts. The common message is decoded by both receivers, while the private message is only decoded by the intended receiver and considered as noise by the other. Optimizing this region is hard due to its complicated structure, and the optimal input distribution remains an open problem to date. However, proper Gaussian codebooks without coded time sharing are common assumptions and achieve the sum-capacity of the GIC in all cases with established sum-capacity. The achievable sum-rate under these assumptions is [10]

$$\begin{aligned}
&\mathrm{C}\left(\frac{\tilde{p}_1}{1+|a_2|^2\tilde{p}_2}\right) + \mathrm{C}\left(\frac{\tilde{p}_2}{1+|a_1|^2\tilde{p}_1}\right) + \\
&\min\Bigg\{\mathrm{C}\left(\frac{(p_1-\tilde{p}_1)+|a_2|^2(p_2-\tilde{p}_2)}{1+\tilde{p}_1+|a_2|^2\tilde{p}_2}\right), \\
&\quad \mathrm{C}\left(\frac{|a_1|^2(p_1-\tilde{p}_1)+(p_2-\tilde{p}_2)}{1+|a_1|^2\tilde{p}_1+\tilde{p}_2}\right), \\
&\quad \mathrm{C}\left(\frac{|a_1|^2(p_1-\tilde{p}_1)}{1+|a_1|^2\tilde{p}_1+\tilde{p}_2}\right) + \mathrm{C}\left(\frac{|a_2|^2(p_2-\tilde{p}_2)}{1+\tilde{p}_1+|a_2|^2\tilde{p}_2}\right), \\
&\quad \mathrm{C}\left(\frac{p_1-\tilde{p}_1}{1+\tilde{p}_1+|a_2|^2\tilde{p}_2}\right) + \mathrm{C}\left(\frac{p_2-\tilde{p}_2}{1+|a_1|^2\tilde{p}_1+\tilde{p}_2}\right)\Bigg\},
\end{aligned} \tag{5}$$

where $p_i \in [0, P_i]$ is the total transmit power of user i and $\tilde{p}_i \in [0, p_i]$ is the power allocated to the private message.

Unconstrained capacity results depend mainly on the channel conditions. Most notably, if $|a_1|^2 \geq 1$ and $|a_2|^2 \geq 1$, the channel is said to have *strong interference* and the capacity region is

$$0 \leq R_i \leq \mathrm{C}(P_i), \quad i = 1,2 \tag{6a}$$

$$R_1 + R_2 \leq \min\{\mathrm{C}(P_1 + |a_2|^2 P_2),\ \mathrm{C}(|a_1|^2 P_1 + P_2)\} \tag{6b}$$

which is achieved by proper Gaussian codewords and jointly decoding x_1 and x_2 at each receiver [11], [12]. The sum-capacity, i.e., the maximal sum-rate $R_1 + R_2$ at which reliable communication is possible, is readily obtained from (6) as

$$C_\Sigma = \min\{\mathrm{C}(P_1) + \mathrm{C}(P_2), \mathrm{C}(P_1 + |a_2|^2 P_2), \mathrm{C}(|a_1|^2 P_1 + P_2)\}. \tag{7}$$

[1]A code is said to be a capacity-achieving PTP code if it achieves the capacity of the Gaussian PTP channel [8, Sect. III].

Apart from the strong interference case, only the sum-capacity is known. The channel has *mixed interference* if only one receiver observes strong interference, i.e., $|a_1|^2 \geq 1$ and $0 < |a_2|^2 \leq 1$ or $0 < |a_1|^2 \leq 1$ and $|a_2|^2 \geq 1$. The sum-capacity for the first case (with $|a_1|^2 \geq 1$) is

$$C_\Sigma = \min\left\{\mathrm{C}(|a_1|^2 P_1 + P_2), \mathrm{C}(P_2) + \mathrm{C}\left(\frac{P_1}{1+|a_2|^2 P_2}\right)\right\}. \tag{8}$$

It is achieved by proper Gaussian codewords, jointly decoding x_1 and x_2 at receiver 2, and treating x_2 as noise at receiver 1. Likewise, the second case is obtained by exchanging the indices in (8) [12], [13]. For $|a_1|^2 < 1$ and $|a_2|^2 < 1$, the channel has *weak interference*. This regime is subdivided by the condition

$$|a_1|(1+|a_2|^2 P_2) + |a_2|(1+|a_1|^2 P_1) \leq 1. \tag{9}$$

If it holds, the interference is *noisy* and the sum-capacity is

$$C_\Sigma = \mathrm{C}\left(\frac{P_1}{1+|a_2|^2 P_2}\right) + \mathrm{C}\left(\frac{P_2}{1+|a_1|^2 P_1}\right), \tag{10}$$

achieved by proper Gaussian inputs and TIN [12]–[15]. Otherwise, the interference is said to be *moderate* and no capacity results are available. Thus, from a throughput perspective, the moderate interference regime is the only one in which HK coding might offer a benefit over PTP codes.

III. Global Energy Efficiency

The GEE of a communication network is defined as the benefit-cost ratio of the total network throughput and the total power necessary to operate the network [16], i.e., for a network with two transmitters,

$$\mathrm{GEE} = W\frac{R_1 + R_2}{\boldsymbol{\mu}^T\boldsymbol{p} + P_c} \quad \left[\frac{\mathrm{bit}}{\mathrm{Joule}}\right], \tag{11}$$

where $R_1, R_2 \geq 0$ are the achievable rates for reliable communication, $\boldsymbol{p} = (p_1, p_2)$ are the transmit powers necessary to achieve these rates, W is the bandwidth, $\boldsymbol{\mu}$ are the power amplifier inefficiencies, and P_c is the total static power consumption of the network. Energy-efficient resource allocation strives to maximize the GEE, i.e., solve the problem

$$\max_{\boldsymbol{p},\boldsymbol{R}} \frac{R_1 + R_2}{\boldsymbol{\mu}^T\boldsymbol{p} + P_c} \quad \mathrm{s.t.} \quad \boldsymbol{R} \in \mathcal{R}(\boldsymbol{p}),\ \boldsymbol{p} \in \mathcal{P} \tag{P1}$$

where $\mathcal{R}(\boldsymbol{p})$ is the achievable rate region of the chosen coding scheme, and $\mathcal{P}$ are the power constraints. For PTP codes, these are $\mathcal{P} = [\boldsymbol{0}, \boldsymbol{P}] = \{\boldsymbol{x} : 0 \leq x_i \leq P_i, \forall i\}$, while for HK coding $\mathcal{P} = \{(p_1, p_2, \tilde{p}_1, \tilde{p}_2) \in \mathbb{R}^4_{\geq 0} : p_i \leq P_i,\ \tilde{p}_i \leq p_i, \text{ for } i = 1,2\}$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, 0, 0)^T$.

Problem (P1) is a non-convex optimization problem that poses two challenges. First, the fractional objective is a non-concave function. Fortunately, fractional programming theory [17] is one of the best developed fields of global optimization, and several solution approaches are at hand. The possibly most popular one is Dinkelbach's Algorithm [18] that solves (P1) as a sequence of auxiliary programs. Given that the objective has concave numerator and convex denominator, these auxiliary problems have a concave objective. The other issue is due to the non-convex rate expressions characterizing $\mathcal{R}(\boldsymbol{p})$ when interfering messages are treated as noise. For example, the right-hand side of (2) is easily verified to be non-concave.

Any optimization problem can be decomposed into an outer and an inner problem by first optimizing over some variables and then over the others. Thus, problem (P1) is equivalent to $\max_{\boldsymbol{p}\in\mathcal{P}} \max_{\boldsymbol{R}\in\mathcal{R}(\boldsymbol{p})} \frac{R_1+R_2}{\boldsymbol{\mu}^T\boldsymbol{p}+P_c}$ and, since the objective's denominator is independent of $\boldsymbol{R}$,

$$\max_{\boldsymbol{p}\in\mathcal{P}} \frac{\max_{\boldsymbol{R}\in\mathcal{R}(\boldsymbol{p})} R_1 + R_2}{\boldsymbol{\mu}^T\boldsymbol{p} + P_c} = \max_{\boldsymbol{p}\in\mathcal{P}} \frac{R_\Sigma(\boldsymbol{p})}{\boldsymbol{\mu}^T\boldsymbol{p} + P_c}. \tag{P2}$$

The numerator of (P2) is the sum-rate which is limited above by the sum-capacity C_Σ. Thus, for constant P_c, the maximum achievable GEE is the solution of (P2) with $R_\Sigma = C_\Sigma$.

Recall from Section II that the sum-capacity is known for all interference regimes except moderate interference. In all cases, it is achievable by PTP codes, which require less complex decoding than HK codes. More complex decoding also requires more energy and, thus, P_c is likely to increase for HK coding. Thus, in cases where both PTP codes and HK coding achieve the sum-capacity, HK coding has a lower GEE than PTP codes. However, for moderate interference, where the sum-capacity is unknown, HK coding might offer a benefit despite its increased decoding complexity. One of the aims of this paper is to verify whether such GEE gains due to HK coding can indeed occur.

The capacity region of GICs constrained to capacity-achieving PTP codes is given in (4) as $\mathcal{C}_{\text{ptp}}$. Then, the maximum achievable GEE when constrained to capacity-achieving PTP codes is the solution of (P1) with $\mathcal{R} = \mathcal{C}_{\text{ptp}}$. Because $\sup_{\boldsymbol{x}\in\bigcup_i \mathcal{D}_i} f(\boldsymbol{x}) = \max_i \sup_{\boldsymbol{x}\in\mathcal{D}_i} f(\boldsymbol{x})$, we can split (P1) into four individual optimization problems, each equivalent to (P1) with $\mathcal{R} = \mathcal{R}_{1,d_1} \cap R_{2,d_2}$ for all different combinations of $d_1, d_2 \in \{\text{TIN}, \text{JD}\}$. These rate regions are exactly the achievable rate regions of the sum-capacity achieving schemes in the four other interference regimes, and, hence, the solution of $(\text{P1})|_{\mathcal{R}=\mathcal{C}_{\text{ptp}}}$ is the maximum of the achievable GEEs in the four other interference regimes.[2] After computing the absolute GEE limit of PTP codes using the algorithms proposed in the following, we can be sure that any gain observed by rate splitting is definitely not achievable by these codes.

A. Strong Interference

In contrast to the other interference regimes, the sum-capacity $C_\Sigma(\boldsymbol{p})$ of the GIC with strong interference is a concave function and, thus, (P2) can be solved as a sequence of convex optimization problems with Dinkelbach's Algorithm. In each iteration, the auxiliary problem

$$\begin{cases} \max\limits_{t,\boldsymbol{p}\in[\boldsymbol{0},\boldsymbol{P}]} & t - \lambda\left(\boldsymbol{\mu}^T\boldsymbol{p} + P_c\right) \\ \text{s.t.} & t \leq \mathrm{C}(p_1) + \mathrm{C}(p_2), \quad t \leq \mathrm{C}(p_1 + |a_2|^2 p_2), \\ & t \leq \mathrm{C}(|a_1|^2 p_1 + p_2). \end{cases} \tag{P3}$$

is solved. The algorithm is stated in Algorithm 1. Please refer to [16], [18] for details on Dinkelbach's Algorithm. Since Dinkelbach's Algorithm has superlinear convergence [17], (P3) is solved very efficiently by Algorithm 1 in combination with a state-of-the-art convex optimization software.

Algorithm 1 Maximum GEE of GIC with Strong Interference

Initialize $\eta > 0$, $i = 0$, $\boldsymbol{p}^0 \in [\boldsymbol{0}, \boldsymbol{P}]$
repeat
  $i \leftarrow i + 1$
  $\lambda^i \leftarrow \frac{C_\Sigma(\boldsymbol{p}^{i-1})}{\boldsymbol{\mu}^T\boldsymbol{p}^{i-1}+P_c}$ with C_Σ as in (7)
  $\boldsymbol{p}^i \leftarrow$ Solution of (P3) with $\lambda = \lambda^i$
until $C_\Sigma(\boldsymbol{p}^i) - \lambda^i(\boldsymbol{\mu}^T\boldsymbol{p}^i + P_c) \leq \eta$
return $\boldsymbol{p}^i$ as the optimal solution

IV. Mixed Monotonic Programming

Consider (P2) with the various non-convex sum-rates discussed in Section II. These are all of the form

$$\sum_i \mathrm{C}\left(\frac{\boldsymbol{a}_i^T\boldsymbol{p}}{1 + \boldsymbol{b}_i^T\boldsymbol{p}}\right), \tag{12}$$

with $\boldsymbol{a}_i, \boldsymbol{b}_i \geq \boldsymbol{0}$, except for the occasional pointwise minimum which we neglect for the moment to ease the exposition. The most popular approach to solve this global optimization problem is to employ the monotonic optimization framework [19], where the main idea is to exploit the monotonicity structure found in many functions. Functions with mixed monotonicity properties are converted to a difference of monotonically increasing functions $g(\boldsymbol{x}) - h(\boldsymbol{x})$ where g, h are increasing functions in $\boldsymbol{x}$, i.e., $\boldsymbol{x}' \geq \boldsymbol{x} \Rightarrow g(\boldsymbol{x}') \geq g(\boldsymbol{x})$. For example, (12) written as a difference of increasing (DI) functions is $\mathrm{C}\big((\boldsymbol{a}_i + \boldsymbol{b}_i)^T\boldsymbol{p}_i\big) - \mathrm{C}\big(\boldsymbol{b}_i^T\boldsymbol{p}_i\big)$.

However, fractional objectives as in (P2) are not easily converted into DI functions. The standard approach to these problems is to combine Dinkelbach's Algorithm with monotonic optimization [7]. This method has the drawbacks that a highly complex auxiliary problem needs to be solved repeatedly with high numerical accuracy to guarantee convergence and that there is no direct relation between the stopping criterion and the distance of the obtained solution to the true optimum. Instead, we propose a more direct branch and bound (BB) approach for functions with mixed monotonic structure that also applies to fractional objectives and has much faster convergence than monotonic optimization.[3]

Consider the optimization problem

$$\max_{\boldsymbol{x}\in\mathcal{F}} \; f(\boldsymbol{x}) \tag{P4}$$

with continuous objective $f : \mathbb{R}^n \mapsto \mathbb{R}$ and convex feasible set $\mathcal{F} \subset [\underline{\boldsymbol{x}}, \bar{\boldsymbol{x}}]$. Assume there exists a function F such that

$$F(\boldsymbol{x}, \boldsymbol{x}) = f(\boldsymbol{x}) \tag{13}$$

and

$$\boldsymbol{x} \leq \boldsymbol{x}' \Rightarrow F(\boldsymbol{x}, \boldsymbol{y}) \leq F(\boldsymbol{x}', \boldsymbol{y}) \tag{14a}$$

$$\boldsymbol{y} \geq \boldsymbol{y}' \Rightarrow F(\boldsymbol{x}, \boldsymbol{y}) \geq F(\boldsymbol{x}, \boldsymbol{y}') \tag{14b}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in [\underline{\boldsymbol{x}}, \bar{\boldsymbol{x}}] \subset \mathbb{R}^n$. A function satisfying (14) is said to be a *mixed monotonic (MM) function* and the optimization problem (P4) is called a *mixed monotonic programming (MMP) problem* if its objective has a MM representation. For example, a MM representation of (12) is $\sum_i \mathrm{C}\Big(\frac{\boldsymbol{a}_i^T\boldsymbol{x}}{1+\boldsymbol{b}_i^T\boldsymbol{y}}\Big)$.

A MMP problem can be solved by a rectangular BB algorithm. Such an algorithm successively partitions the feasible set $\mathcal{F}$ into boxes by a bisection of ratio $\frac{1}{2}$. That is, given a

[2] These are strong, noisy, and two cases of mixed interference.

[3] The faster convergence of MMP can be verified by showing that the bounds obtained by MMP are always tighter than those of monotonic programming for functions of the form (12).

box $\mathcal{M} = [\boldsymbol{r}, \boldsymbol{s}]$, the partition sets that replace this box are the subrectangles

$$\begin{aligned} \mathcal{M}_- &= \{\boldsymbol{x} : r_j \le x_j \le v_j,\ r_i \le x_i \le s_i\ (i \neq j)\} \\ \mathcal{M}_+ &= \{\boldsymbol{x} : v_j \le x_j \le s_j,\ r_i \le x_i \le s_i\ (i \neq j)\} \end{aligned} \tag{15}$$

where $j \in \arg\max_j s_j - r_j$ is a longest side of $\mathcal{M}$ and $v_j = \frac{s_j + r_j}{2}$ defines the hyperplane along which $\mathcal{M}$ is divided. The bisection is started from an initial box $[\underline{\boldsymbol{x}}, \bar{\boldsymbol{x}}]$ that contains the feasbile set $\mathcal{F}$. For each box $\mathcal{M}$, the algorithm computes an upper bound $\beta(\mathcal{M})$ for the objective values in $\mathcal{M}$. It terminates if the upper bound of a box is close to the objective value of some previously found feasible point. The mixed monotonic structure of the objective function facilitates computation of the upper bound $\beta(\mathcal{M})$. From (14), we have

$$\max_{\boldsymbol{x} \in \mathcal{M} \cap \mathcal{F}} f(\boldsymbol{x}) \overset{(13)}{=} \max_{\boldsymbol{x} \in [\boldsymbol{r}, \boldsymbol{s}] \cap \mathcal{F}} F(\boldsymbol{x}, \boldsymbol{x}) \overset{(14a)}{\le} \max_{\boldsymbol{x} \in [\boldsymbol{r}, \boldsymbol{s}]} F(\boldsymbol{s}, \boldsymbol{x}) \overset{(14b)}{=} F(\boldsymbol{s}, \boldsymbol{r}),$$

and, thus, $\beta(\mathcal{M}) = F(\boldsymbol{s}, \boldsymbol{r})$. The complete algorithm is stated in Algorithm 2. Its convergence is established below.

Proposition 1: For every $\varepsilon > 0$, Algorithm 2 converges in a finite number of iterations to a point with objective value within an ε-region of the globally optimal value of (P4).

Proof: By virtue of [19, Prop. 6.1], Algorithm 2 is convergent if the subdivision is exhaustive and bounding is consistent. For problem (P4), we say that bounding is consistent with branching if

$$F(\boldsymbol{s}, \boldsymbol{r}) - \max\{f(\boldsymbol{x}) : \boldsymbol{x} \in [\boldsymbol{r}, \boldsymbol{s}] \cap \mathcal{F}\} \to 0 \tag{16}$$

as $\|\boldsymbol{s} - \boldsymbol{r}\| \to 0$. Let $\tilde{\boldsymbol{x}}$ be the maximizer of $f(\boldsymbol{x})$ over $[\boldsymbol{r}, \boldsymbol{s}] \cap \mathcal{F}$. Also, as $\|\boldsymbol{s} - \boldsymbol{r}\| \to 0$, $\tilde{\boldsymbol{x}}$ is the common limit of $\boldsymbol{r}$ and $\boldsymbol{s}$ and $\tilde{\boldsymbol{x}} \in \mathcal{F}$ due to line 7 in Algorithm 2. Thus, $F(\boldsymbol{s}, \boldsymbol{r}) - f(\tilde{\boldsymbol{x}}) \to F(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) - f(\tilde{\boldsymbol{x}}) = 0$. Finally, the bisection in (15) is exhaustive [19, Cor. 6.2], and, thus, $\|\boldsymbol{s}^k - \boldsymbol{r}^k\| \to 0$ as $k \to \infty$. ■

Algorithm 2 Mixed Monotonic Programming

1: Initialize $\mathscr{R}_0 = \{[\underline{\boldsymbol{x}}, \bar{\boldsymbol{x}}]\}$, $\gamma = -\infty$, $k = 0$, $\varepsilon > 0$
2: **repeat**
3: Select $[\boldsymbol{r}^k, \boldsymbol{s}^k] \in \arg\max\{F(\boldsymbol{s}, \boldsymbol{r}) \,|\, [\boldsymbol{r}, \boldsymbol{s}] \in \mathscr{R}_k\}$
4: Select $j_k \in \arg\max_j s_j^k - r_j^k$
5: Compute $\mathscr{P}_k = \{\mathcal{M}_-^k, \mathcal{M}_+^k\}$ as in (15) with $v_{j_k} = \frac{s_j + r_j}{2}$
6: **for all** $[\boldsymbol{r}, \boldsymbol{s}] \in \mathscr{P}_k$ **do**
7: **if** $[\boldsymbol{r}, \boldsymbol{s}] \cap \mathcal{F} \neq \emptyset$ **and** $F(\boldsymbol{s}, \boldsymbol{r}) > \gamma + \varepsilon$ **then**
8: Add $[\boldsymbol{r}, \boldsymbol{s}]$ to $\mathscr{S}_k$
9: Find $\boldsymbol{x} \in \mathcal{F} \cap [\boldsymbol{r}, \boldsymbol{s}]$
10: **if** $f(\boldsymbol{x}) > \gamma$ **then**
11: $\bar{\boldsymbol{x}} \leftarrow \boldsymbol{x}$
12: $\gamma \leftarrow f(\boldsymbol{x})$
13: **end if**
14: **end if**
15: **end for**
16: $\mathscr{R}_{k+1} \leftarrow \mathscr{S}_k \cup \mathscr{R}_k \setminus \{[\boldsymbol{r}^k, \boldsymbol{s}^k]\}$
17: $k \leftarrow k + 1$
18: **until** $\mathscr{R}_k = \emptyset$
19: **return** $\bar{\boldsymbol{x}}$ as the optimal solution

A. Application to GICs

We now apply Algorithm 2 to the solution of (P2) for all interference regimes except the strong interference case, which is already solved by Algorithm 1. First, identify $\mathcal{F} = \mathcal{P}$ and observe that for all coding schemes except HK coding, every box $\mathcal{M}$ generated by Algorithm 2 is $\mathcal{M} \subset \mathcal{P}$. Thus, the first condition in line 7 is always satisfied and the feasibility problem in line 9 is trivial. Instead, for HK coding let the optimization variables be ordered as $\boldsymbol{x} = (\boldsymbol{p}, \tilde{\boldsymbol{p}})$. Then, for a box $[\boldsymbol{r}, \boldsymbol{s}] \times [\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{s}}]$, the first condition in line 7 is equivalent to checking $\boldsymbol{s} \ge \tilde{\boldsymbol{r}}$, and a solution to the feasibility problem in line 9 is $(\boldsymbol{s}, \tilde{\boldsymbol{r}})$.

The objective function in all interference cases is $f(\boldsymbol{p}) = \frac{R_\Sigma(\boldsymbol{p})}{\boldsymbol{\mu}^T \boldsymbol{p} + P_c}$ with $R_\Sigma(\boldsymbol{p})$ being one of the sum-rates/-capacities defined in Section II. Let $F_\Sigma(\boldsymbol{x}, \boldsymbol{y})$ be a MM representation of $R_\Sigma(\boldsymbol{p})$. Then, a suitable MM representation of $f(\boldsymbol{p})$ is $F(\boldsymbol{x}, \boldsymbol{y}) = \frac{F_\Sigma(\boldsymbol{x}, \boldsymbol{y})}{\boldsymbol{\mu}^T \boldsymbol{y} + P_c}$. Thus, all that is required to solve the GEE maximization problem with Algorithm 2 is an MM representation for the achievable sum-rates. These are

$$F_\Sigma(\boldsymbol{x}, \boldsymbol{y}) = \min\left\{ \mathrm{C}(|a_1|^2 x_1 + x_2), \mathrm{C}(x_2) + \mathrm{C}\left(\frac{x_1}{1 + |a_2|^2 y_2}\right) \right\}$$

for mixed interference,

$$F_\Sigma(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{C}\left(\frac{x_1}{1 + |a_2|^2 y_2}\right) + \mathrm{C}\left(\frac{x_2}{1 + |a_1|^2 y_1}\right)$$

for noisy interference, and

$$\begin{aligned} F_\Sigma(\boldsymbol{x}, \boldsymbol{y}) = {} & \mathrm{C}\left(\frac{\tilde{x}_1}{1 + |a_2|^2 \tilde{y}_2}\right) + \mathrm{C}\left(\frac{\tilde{x}_2}{1 + |a_1|^2 \tilde{y}_1}\right) + \\ \min\Bigg\{ & \mathrm{C}\left(\frac{(x_1 - \tilde{y}_1) + |a_2|^2 (x_2 - \tilde{y}_2)}{1 + \tilde{y}_1 + |a_2|^2 \tilde{y}_2}\right), \\ & \mathrm{C}\left(\frac{|a_1|^2 (x_1 - \tilde{y}_1) + (x_2 - \tilde{y}_2)}{1 + |a_1|^2 \tilde{y}_1 + \tilde{y}_2}\right), \\ & \mathrm{C}\left(\frac{|a_1|^2 (x_1 - \tilde{y}_1)}{1 + |a_1|^2 \tilde{y}_1 + \tilde{y}_2}\right) + \mathrm{C}\left(\frac{|a_2|^2 (x_2 - \tilde{y}_2)}{1 + \tilde{y}_1 + |a_2|^2 \tilde{y}_2}\right), \\ & \mathrm{C}\left(\frac{x_1 - \tilde{y}_1}{1 + \tilde{y}_1 + |a_2|^2 \tilde{y}_2}\right) + \mathrm{C}\left(\frac{x_2 - \tilde{y}_2}{1 + |a_1|^2 \tilde{y}_1 + \tilde{y}_2}\right) \Bigg\} \end{aligned}$$

for HK coding. In all cases, verifying (14) is straightforward.

V. Numerical Evaluation

We consider a wireless interference network in which two single-antenna senders are placed randomly with uniform distribution in a $1\,\mathrm{km} \times 2\,\mathrm{km}$ rectangular area and communicate with two single-antenna receivers placed at coordinates $(-0.5, 0)$ km and $(0.5, 0)$ km. The path-loss is modeled according to [20], with power decay factor 3.5 and carrier frequency 1.8 GHz, while small-scale fading effects are modeled as i.i.d. proper Gaussian random variates. The communication bandwidth is 180 kHz and the noise spectral density is -174 dBm/Hz. Each receiver has a 3 dB noise figure and a power amplifier inefficiency $\mu = 4$. Static power consumption is $P_c = 1\,\mathrm{W}$ and all senders have the same maximum transmit power $P_{\max}$.

Two different scenarios are considered: One where the transmitters are dropped in the whole area and always associated with the same receiver, and one where each transmitter is placed such that it is geographically closer to its respective receiver. The latter resembles an uplink multi-cell scenario where the receivers represent base stations. Table I shows the empirical probability of the resulting channel being in a specific interference regime. It is noteworthy that, while with arbitrary user placement the distribution is approximately uniform,[4] the

[4]From this point of view, the moderate and noisy interference regimes should be considered together since the differentiation between these two depends on the transmit power.

TABLE I
EMPIRICAL PROBABILITY OF INTERFERENCE REGIMES

	$P_{\max}$	Strong	Mixed	Weak Moderate	Weak Noisy
Arbitrary	0 dBm	34.8 %	31.5 %	11.0 %	22.7 %
	15 dBm			21.6 %	12.1 %
	30 dBm			32.3 %	1.4 %
Multi Cell	0 dBm	0.3 %	9.3 %	11.3 %	79.1 %
	15 dBm			44.1 %	46.3 %
	30 dBm			85.3 %	5.1 %

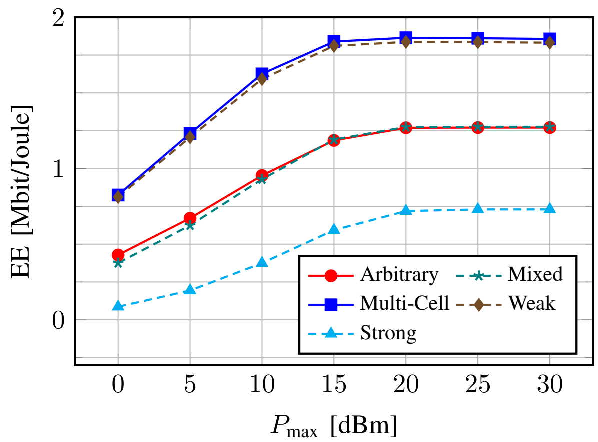

Fig. 1. GEE of GIC with arbitrary and multi-cell user placement. Averaged over 1,000+ i.i.d. channel realizations.

weak interference regime clearly dominates for the multi-cell scenario. Especially for realistic transmit powers, moderate interference clearly dominates. Thus, the only interference regime where rate splitting might provide a benefit over PTP codes is definitely non-negligible.

Numerical results for both user placements obtained by Algorithms 1 and 2 with tolerance $\varepsilon = 0.01$ are displayed in Fig. 1. In addition, the GEEs for the individual interference regimes are shown. It can be observed that weak interference achieves the highest GEE, followed by mixed interference and strong interference. As could be expected from Table I, the multi-cell scenario achieves a similar GEE as weak interference, and the arbitrary scenario is approximately in the middle of the three interference regimes. The slightly better performance of the multi-cell scenario compared to the weak interference regime is due to statistical effects from not completely overlapping sets of channel realizations.

The maximum GEE in the moderate interference regime is unknown. In all other regimes, it is achieved by PTP codes. To study whether this is also the case under moderate interference, we compare the maximum GEE achievable with PTP codes with the GEE of HK coding with proper Gaussian codebooks without coded time sharing. In 9 out of 5,390 analyzed channel realizations that fall within the moderate interference regime, HK coding achieves a numerically significant gain when ignoring the different circuit power consumptions P_c. This implies that there can still be gains if P_c is modeled as being higher for HK than for PTP as long as the difference in P_c is not too large. The maximal observed gain over PTP codes is 0.11 bit/J/Hz (4.44 %) which amounts to 19.71 kbit/J. Having observed that HK coding can be beneficial in terms of EE, it should be studied in future research whether more general HK coding with coded time sharing and codebooks other than proper Gaussian can bring higher gains and/or gains in a larger number of channel realizations.

VI. Conclusions

We have shown that Gaussian PTP codes are the most energy-efficient in the GIC except for moderate interference, where HK coding can achieve a higher GEE. The numerical assessment is facilitated by mixed monotonic programming, a novel global optimization framework that is easily applicable to a wide class of functions and has faster convergence than state-of-the-art solutions based on monotonic optimization. This framework is also suitable to assess possible gains of more general HK coding, which is left for future work.

References

[1] S. Buzzi, C.-L. I, T. E. Klein, H. V. Poor, C. Yang, and A. Zappone, “A survey of energy-efficient techniques for 5G networks and challenges ahead,” *IEEE J. Sel. Areas Commun.*, vol. 34, no. 4, pp. 697–709, 2016.

[2] A. B. Carleial, “Interference channels,” *IEEE Trans. Inf. Theory*, vol. 24, no. 1, pp. 60–70, Jan. 1978.

[3] S. Yang, M. Kobayashi, D. Gesbert, and X. Yi, “Degrees of freedom of time correlated MISO broadcast channel with delayed CSIT,” *IEEE Trans. Inf. Theory*, vol. 59, no. 1, pp. 315–328, Jan. 2013.

[4] B. Clerckx, H. Joudeh, C. Hao, M. Dai, and B. Rassouli, “Rate splitting for MIMO wireless networks: a promising PHY-layer strategy for LTE evolution,” *IEEE Commun. Mag.*, vol. 54, no. 5, pp. 98–105, May 2016.

[5] T. Han and K. Kobayashi, “A new achievable rate region for the interference channel,” *IEEE Trans. Inf. Theory*, vol. 27, no. 1, 1981.

[6] R. H. Etkin, D. N. C. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.

[7] A. Zappone, E. Björnson, L. Sanguinetti, and E. A. Jorswieck, “Globally optimal energy-efficient power control and receiver design in wireless networks,” *IEEE Trans. Signal Process.*, vol. 65, no. 11, Jun. 2017.

[8] F. Baccelli, A. El Gamal, and D. N. C. Tse, “Interference networks with point-to-point codes,” *IEEE Trans. Inf. Theory*, vol. 57, no. 5, pp. 2582–2596, Apr. 2011.

[9] B. Bandemer, A. El Gamal, and Y.-H. Kim, “Optimal achievable rates for interference networks with random codes,” *IEEE Trans. Inf. Theory*, vol. 61, no. 12, pp. 6536–6549, Dec. 2015.

[10] D. Tuninetti and Y. Weng, “On the Han-Kobayashi achievable region for Gaussian interference channels,” in *Proc. IEEE ISIT*, Jul. 2008.

[11] H. Sato, “The capacity of the Gaussian interference channel under strong interference (corresp.),” *IEEE Trans. Inf. Theory*, vol. 27, no. 6, pp. 786–788, Nov. 1981.

[12] X. Shang, B. Chen, G. Kramer, and H. V. Poor, “Capacity regions and sum-rate capacities of vector Gaussian interference channels,” *IEEE Trans. Inf. Theory*, vol. 56, no. 10, pp. 5030–5044, Oct. 2010.

[13] A. S. Motahari and A. K. Khandani, “Capacity bounds for the Gaussian interference channel,” *IEEE Trans. Inf. Theory*, vol. 55, no. 2, Feb. 2009.

[14] V. S. Annapureddy and V. V. Veeravalli, “Gaussian interference networks: Sum capacity in the low-interference regime and new outer bounds on the capacity region,” *IEEE Trans. Inf. Theory*, vol. 55, no. 7, Jul. 2009.

[15] X. Shang, G. Kramer, and B. Chen, “A new outer bound and the noisy-interference sum-rate capacity for Gaussian interference channels,” *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 689–699, Feb. 2009.

[16] A. Zappone and E. A. Jorswieck, *Energy Efficiency in Wireless Networks via Fractional Programming Theory*. Now Publishers, 2015.

[17] S. Schaible, “Fractional programming,” in *Handbook of global optimization*, R. Horst and P. M. Pardalos, Eds. Kluwer, 1995.

[18] W. Dinkelbach, “On nonlinear fractional programming,” *Management Science*, vol. 13, no. 7, pp. 492–498, Mar. 1967.

[19] H. Tuy, *Convex Analysis and Global Optimization*. Springer, 2016.

[20] G. Calcev *et al.*, “A wideband spatial channel model for system-wide simulations,” *IEEE Trans. Veh. Technol.*, vol. 56, no. 2, March 2007.