GLOBAL ENERGY EFFICIENCY MAXIMIZATION IN NON-ORTHOGONAL INTERFERENCE NETWORKS

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ABSTRACT

Energy efficient resource allocation in interference networks is a challenging global optimization problem. The main issue is that the computational complexity grows exponentially in the number of variables. In general, resource allocation in interference networks requires optimizing jointly over achievable rates and transmit powers. However, close scrutiny reveals that the non-convexity stems mostly from the powers while the problem is linear in the rates. Conventional global optimization frameworks treat all variables as non-convex and require complicated, problem specific decomposition approaches to exploit the convexity in some variables. Another issue specific to energy-efficient resource allocation is that these frameworks are unable to deal directly with fractional objectives. The usual approach is to use Dinkelbach's algorithm which requires the solution of a sequence of auxiliary global optimization problems. This increases the computational complexity significantly. To overcome these challenges, we develop an algorithm that inherently treats fractional objectives and differentiates between convex and non-convex variables, preserving the polynomial complexity in the number of convex variables. The numerical results show a speed-up of almost four orders of magnitude over Dinkelbach's algorithm for global fractional programs.

Index Terms— Global optimization, Resource allocation, Energy Efficiency, Interference Networks

1. INTRODUCTION

The global energy efficiency is the most widely used metric to measure the network energy efficiency, a key performance metric in 5G and beyond networks [1, 2]. It is defined as the benefit-cost ratio of the total network throughput and the associated power consumption, i.e., $\mathrm{GEE} = \frac{\sum_k R_k}{\boldsymbol{\phi}^T \boldsymbol{p} + P_c}$, where R_k is the achievable rate of link k, $\boldsymbol{p}$ are the transmission powers necessary to achieve these rates, $\boldsymbol{\phi} \geq 1$ are the power amplifier inefficiencies and P_c is the total circuit power necessary to operate the network. The corresponding resource allocation problem in many Gaussian interference networks is

$$\begin{cases} \max\limits_{\boldsymbol{p},\boldsymbol{R}} & \dfrac{\sum_k R_k}{\boldsymbol{\phi}^T \boldsymbol{p} + P_c} \\ \text{s.t.} & \boldsymbol{a}_i^T \boldsymbol{R} \leq \log\left(1 + \dfrac{\boldsymbol{b}_i^T \boldsymbol{p}}{\boldsymbol{c}_i^T \boldsymbol{p} + \sigma_i}\right), \quad i = 1, \dots, n \\ & \boldsymbol{R} \geq 0, \quad \boldsymbol{p} \in [\boldsymbol{0}, \boldsymbol{P}] \end{cases} \tag{P1}$$

for positive vectors $\boldsymbol{a}_i, \boldsymbol{b}_i, \boldsymbol{c}_i \geq 0$, $i = 1, \dots, n$. This is a challenging global optimization problem due the fractional objective and the non-convex right-hand sides of the constraints. Hence, in general, the computational complexity of solving this problem grows exponentially in the number of variables [3]. A close inspection, however, reveals that the non-convexity of (P1) stems only from the powers $\boldsymbol{p}$, since, for fixed $\boldsymbol{p}$, (P1) is linear in $\boldsymbol{R}$.

The most popular solution approaches for global resource allocation problems are monotonic optimization [4] and DC programming [5]. Both frameworks treat all variables as global variables which results in unnecessary high computational complexity if, like (P1), the problem at hand is convex in some variables. Moreover, the fractional objective of (P1) can not be handled directly by these frameworks. Instead, Dinkelbach's algorithm [6,7] is used where the original problem is transformed into an auxiliary problem, which is then solved several times with one of these frameworks [8]. However, this approach has the drawbacks that convergence to the optimal solution is only guaranteed if the auxiliary problem is solved exactly, that the stopping criterion is unrelated to the distance of the obtained approximate optimal value to the true optimum, and that the auxiliary problem needs to be solved several times. Especially, the last issue further increases computational complexity significantly.

Instead, in this paper we develop a novel algorithm that completely avoids these issues. Specifically, it preserves the polynomial computational complexity in the convex variables and inherently supports fractional objectives without the need of a Dinkelbach-like outer algorithm. It also avoids another often neglected issue with conventional global optimization approaches that originates from (ε, η)-approximate feasibility. Every numerical optimization procedure has to be stopped at some point which is expected to be close enough to the optimal solution. This point should at least be approximately feasible and provide an objective value sufficiently close to the true optimal value. Mathematically, a point $\bar{\boldsymbol{x}}$ is accepted as the solution of the continuous optimization problem

$$\max\{f(\boldsymbol{x}) | \boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}], g_i(\boldsymbol{x}) \leq 0, i = 1, \dots, m\} \tag{P2}$$

if it satisfies, for some $\varepsilon, \eta > 0$, $g_i(\bar{\boldsymbol{x}}) \leq \varepsilon$ for all $i = 1, \dots, m$ and $f(\bar{\boldsymbol{x}}) \geq f(\boldsymbol{x}) - \eta$ for all feasible $\boldsymbol{x}$. This approach tends towards the optimal solution as $\varepsilon, \eta \to 0$ and, thus, $\bar{\boldsymbol{x}}(\varepsilon)$ should be close to the optimal solution for some sufficiently small $\varepsilon < \varepsilon_0$. However, in general this ε_0 is unknown and hard to determine. The effects of choosing $\varepsilon > \varepsilon_0$ too large range from convergence issues to creating new (isolated) feasible points that might have much higher objective value than the true optimum and, thus, lead to completely wrong results [9–11].

These issues are remedied by the essential feasibility concept that is implemented by the successive incumbent transcending (SIT) scheme [10]. The core idea behind ε-essential feasibility is to shrink

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the feasible set by an infinitesimal amount and, thus, to eliminate every isolated feasible point that might lead to convergence problems and numerically unstable solutions. The SIT scheme then solves a sequence of feasibility problems with a branch and bound (BB) procedure. This results in a numerically much stabler procedure than could be obtained using classical monotonic or DC programming algorithms. Moreover, the SIT approach always provides a good feasible solution even if stopped prematurely. Instead, conventional algorithms usually outer approximate the solution rendering intermediate solutions almost useless (because they are infeasbile).

Related work & Contributions: The SIT approach was developed by Hoang Tuy in [9–11]. We extend this algorithm to fractional objectives, multiple constraints, and differentiate between convex and non-convex variable such that the polynomial complexity in the number of convex variables is preserved.

Energy-efficient resource allocation is usually done under orthogonality assumptions where the non-convexity already is within the objective, e.g., [8, 12, 13]. Instead, in [14, 15] we consider throughput optimal resource allocation for multi user decoding. In this work, we extend the SIT based approach from [15] to energy efficiency maximization without the need of Dinkelbach's iterative algorithm [6, 7].

Notation & Preliminaries: A vector $\boldsymbol{x} \in \mathbb{R}^n$ with components $(x_1, \dots, x_n)$ is said to *dominate* another vector $\boldsymbol{y} \in \mathbb{R}^n$, i.e., $\boldsymbol{y} \leq \boldsymbol{x}$, if $y_i \leq x_i$ for all $i = 1, \dots, n$. For $\boldsymbol{a} \leq \boldsymbol{b}$, the set $[\boldsymbol{a}, \boldsymbol{b}] = \{\boldsymbol{x} \,|\, \boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b}\}$ is called a *box*. A function $f : \mathbb{R}^n_{\geq 0} \mapsto \mathbb{R}$ is *increasing* if $f(\boldsymbol{x}') \leq f(\boldsymbol{x})$ whenever $\boldsymbol{x}' \leq \boldsymbol{x}$, and *decreasing* if $-f$ is increasing. A *common minimizer (maximizer)* of the functions $f_1(\boldsymbol{x}), \dots, f_n(\boldsymbol{x})$ over the set $\mathcal{X}$ is any $\boldsymbol{x}^*$ that satisfies $\boldsymbol{x}^* \in \bigcap_{i=1}^n \arg\min_{\boldsymbol{x}\in\mathcal{X}} f_i(\boldsymbol{x})$ ($\boldsymbol{x}^* \in \bigcap_{i=1}^n \arg\max_{\boldsymbol{x}\in\mathcal{X}} f_i(\boldsymbol{x})$). Consider, for example, the optimization problem (P2). Its optimal solution is $\boldsymbol{x}^*$ and the optimal value is $v(\text{P2}) = f(\boldsymbol{x}^*)$.

A set $\mathcal{G} \subseteq \mathbb{R}^n_{\geq 0}$ is said to be *normal* if for $0 \leq \boldsymbol{x}' \leq \boldsymbol{x}, \boldsymbol{x} \in \mathcal{G} \Rightarrow \boldsymbol{x}' \in \mathcal{G}$, and *normal in a box* $[\boldsymbol{a}, \boldsymbol{b}]$ if the previous implication only holds for $\boldsymbol{a} \leq \boldsymbol{x}' \leq \boldsymbol{x} \leq \boldsymbol{b}$. A set $\mathcal{H} \subseteq \mathbb{R}^n_{\geq 0}$ is called *conormal* if $\boldsymbol{x} + \mathbb{R}^n_{\geq 0} \subseteq \mathcal{H}$ whenever $\boldsymbol{x} \in \mathcal{H}$, and *conormal in a box* $[\boldsymbol{a}, \boldsymbol{b}]$ if for $\boldsymbol{b} \geq \boldsymbol{x}' \geq \boldsymbol{x} \geq \boldsymbol{a}, \boldsymbol{x} \in \mathcal{H} \Rightarrow \boldsymbol{x}' \in \mathcal{H}$ [11, Sec. 11.1.1]. Let $\mathcal{A} \subseteq \mathbb{R}^n$. Then, $\mathcal{A}$ is robust if it satisfies $\mathcal{F}^* = \operatorname{cl}(\operatorname{int} \mathcal{F})$ where cl and int denote the closure and interior, respectively. This is equivalent to saying that $\mathcal{A}$ has no isolated points, i.e., a single point at the center of a ball containing no other feasible points. Finally, $\operatorname{diam} \mathcal{A}$ is the *diameter* of $\mathcal{A}$, i.e., the maximum distance between two points in $\mathcal{A}$; and $\mathcal{A}_{\tilde{\boldsymbol{x}}} = \{\boldsymbol{y} | (\tilde{\boldsymbol{x}}, \boldsymbol{y}) \in \mathcal{D}\}$ is called the $\tilde{\boldsymbol{x}}$*-section* of $\mathcal{A}$.

2. PROBLEM STATEMENT

We can recast (P1) into the global optimization problem

$$\begin{cases} \max\limits_{(\boldsymbol{x},\boldsymbol{\xi})\in\mathcal{X}\times\Xi} & \dfrac{f^+(\boldsymbol{x},\boldsymbol{\xi})}{f^-(\boldsymbol{x},\boldsymbol{\xi})} \\ \text{s.t.} & g_i^+(\boldsymbol{x},\boldsymbol{\xi}) - g_i^-(\boldsymbol{x}) \leq 0, \ i = 1, \dots, m \end{cases} \tag{P3}$$

with convex variables $\boldsymbol{\xi}$ and non-convex variables $\boldsymbol{x}$. The equivalence to problem (P1) is easily established by identifying $\boldsymbol{R}$ as $\boldsymbol{\xi}$ and $\boldsymbol{p}$ as $\boldsymbol{x}$, $\mathcal{X} = [\boldsymbol{0}, \boldsymbol{\infty})$ and $\Xi = [\boldsymbol{0}, \boldsymbol{P}]$, $f^+(\boldsymbol{p}, \boldsymbol{R}) := \sum_k R_k$ and $f^-(\boldsymbol{p}, \boldsymbol{R}) := \boldsymbol{\phi}^T \boldsymbol{p} + P_c$, and

$$\begin{aligned} g_i^+(\boldsymbol{p}, \boldsymbol{R}) &:= \boldsymbol{a}_i^T \boldsymbol{R} + \log\Big(\boldsymbol{c}_i^T \boldsymbol{p} + \sigma_i\Big) \\ g_i^-(\boldsymbol{p}) &:= \log\Big(\Big(\boldsymbol{b}_i^T + \boldsymbol{c}_i^T\Big)\boldsymbol{p} + \sigma_i\Big). \end{aligned} \tag{1}$$

The goal of this paper is to design a numerically stable BB procedure to solve (P3) with polynomial complexity in $\boldsymbol{\xi}$ and exponential complexity in $\boldsymbol{x}$. In general, we require (P3) to satisfy the following technical conditions. The set Ξ is a closed convex set and the functions $\{g_i^-(\boldsymbol{x})\}$ have a common maximizer over every box $[\underline{\boldsymbol{x}}, \bar{\boldsymbol{x}}] \subseteq \mathcal{M}_0$ with $\mathcal{M}_0$ being a rectangular set enclosing $\mathcal{X}$. The functions $f^-(\boldsymbol{x}, \boldsymbol{\xi})$, $g_i^+(\boldsymbol{x}, \boldsymbol{\xi})$, $i = 1, \dots, m$, are lower semi-continuous (l.s.c.), and $f^+(\boldsymbol{x}, \boldsymbol{\xi})$, $g_i^-(\boldsymbol{x})$, $i = 1, \dots, m$, are upper semi-continuous. Without loss of generality, $f^-(\boldsymbol{x}, \boldsymbol{\xi}) > 0$. Further, let each function of $(\boldsymbol{x}, \boldsymbol{\xi})$ be separable in the sense that $h(\boldsymbol{x}, \boldsymbol{\xi}) = h_x(\boldsymbol{x}) + h_\xi(\boldsymbol{\xi})$. Let the functions $\gamma f_\xi^-(\boldsymbol{\xi}) - f_\xi^+(\boldsymbol{\xi})$, $g_{1,\xi}^+(\boldsymbol{\xi}), \dots, g_{m,\xi}^+(\boldsymbol{\xi})$ be convex in $\boldsymbol{\xi}$ for all γ (to be defined later), and let $\gamma f_x^-(\boldsymbol{x}) - f_x^+(\boldsymbol{x})$, $g_{1,x}^+(\boldsymbol{x}), \dots, g_{m,x}^+(\boldsymbol{x})$ have a common minimizer over $\mathcal{X} \cap \mathcal{M}$ for every box $\mathcal{M} \subseteq \mathcal{M}_0$ and all γ. Finally, let the function $\gamma f_x^-(\boldsymbol{x}) - f_x^+(\boldsymbol{x})$ be either increasing for all γ with $\mathcal{X}$ being a closed normal set in some box, or decreasing for all γ with $\mathcal{X}$ being a closed conormal set in some box.

Several conditions on the objective depend on the constant γ which will hold the current best known value in the developed algorithm. Observe that the only relevant property of γ is its sign and whether it may change during the algorithm. For example, the function $\gamma f_\xi^-(\boldsymbol{\xi}) - f_\xi^+(\boldsymbol{\xi})$ is convex if $f_\xi^+(\boldsymbol{\xi})$ is concave and $\gamma f_\xi^-(\boldsymbol{\xi})$ is convex. This is the case if $\gamma \geq 0$ and $f_\xi^-(\boldsymbol{\xi})$ is convex or if $\gamma \leq 0$ and $f_\xi^-(\boldsymbol{\xi})$ is concave. Thus, in most cases we should ensure that the sign of γ is constant. In general, γ may take values between some γ_0 and $v(P3) + \eta$ for some small $\eta > 0$. The lower end of the range γ_0 is either the objective value of (P3) for some preliminary known nonisolated feasible point $(\boldsymbol{x}, \boldsymbol{\xi})$ or an arbitrary value satisfying $\gamma_0 \leq \frac{f^+(\boldsymbol{x},\boldsymbol{\xi})}{f^-(\boldsymbol{x},\boldsymbol{\xi})}$ for all feasible $(\boldsymbol{x}, \boldsymbol{\xi})$. This implies, e.g., that γ is non-negative if $f^+(\boldsymbol{x}, \boldsymbol{\xi})$ is non-negative. Otherwise, it might be necessary to find a nonisolated feasible point such that $f^+(\boldsymbol{x}, \boldsymbol{\xi}) \geq 0$ or transform the problem.

3. ROBUST GLOBAL RESOURCE ALLOCATION

In [15] and the current paper, we employ the SIT approach [11, Sect. 7.5.1] for the first time to solve a resource allocation problem in wireless communications and obtain an essential (ε, η)-optimal solution of (P3). That is, a solution that satisfies $\frac{f^+(\boldsymbol{x}^*,\boldsymbol{\xi}^*)}{f^-(\boldsymbol{x}^*,\boldsymbol{\xi}^*)} + \eta \geq \frac{f^+(\boldsymbol{x},\boldsymbol{\xi})}{f^-(\boldsymbol{x},\boldsymbol{\xi})}$ for some $\varepsilon, \eta > 0$ and all $(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathcal{X} \times \Xi$ such that $g_i^+(\boldsymbol{x}, \boldsymbol{\xi}) - g_i^-(\boldsymbol{x}) \leq -\varepsilon$, $i = 1, \dots, m$. Clearly, for $\varepsilon, \eta \to 0$ an essential (ε, η)-optimal solution is a nonisolated feasible point which is optimal. The core problem in the SIT scheme is, given a real number γ, to check whether (P3) has a nonisolated feasible solution $(\boldsymbol{x}, \boldsymbol{\xi})$ satisfying $f(\boldsymbol{x}, \boldsymbol{\xi}) \geq \gamma$, or, else, establish that no such ε-essential feasible $(\boldsymbol{x}, \boldsymbol{\xi})$ exists. Given that this subproblem is solved within finitely many steps, the algorithm converges to the global optimal solution in a finite number of steps.

Consider the optimization problem

$$\begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{\xi})\in\mathcal{X}\times\Xi} & \max\limits_{i=1,2,\dots,m} \big(g_i^+(\boldsymbol{x},\boldsymbol{\xi}) - g_i^-(\boldsymbol{x})\big) \\ \text{s.t.} & \dfrac{f^+(\boldsymbol{x},\boldsymbol{\xi})}{f^-(\boldsymbol{x},\boldsymbol{\xi})} \geq \gamma \end{cases} \tag{P4}$$

where we exchanged objective and constraints of (P3). Observe that the constraint is equivalent to $\gamma f^-(\boldsymbol{x}, \boldsymbol{\xi}) - f^+(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0$ since $f^-(\boldsymbol{x}, \boldsymbol{\xi}) > 0$ by assumption. The following proposition establishes that the feasibility problem in the SIT scheme is equivalent to solving (P4).

Proposition 1 (adapted from [11, Prop. 7.13]). *For every $\varepsilon > 0$, the ε-essential optimal value of* (P3) *is less than γ if the optimal value of* (P4) *is greater than $-\varepsilon$.*

In contrast to (P3), the feasible set of (P4) is robust and it can be solved efficiently using an adaptive BB procedure [11, Prop. 6.2]. This is formally established in the following lemma.

Lemma 1. *The feasible set of* (P4) *is robust, i.e., it does not contain any isolated points.*

Proof. Let $\mathcal{D}$ be the feasible set of (P4), i.e., $\mathcal{D} = \{\boldsymbol{x} \in \mathcal{X}, \boldsymbol{\xi} \in \Xi : f_\xi(\boldsymbol{\xi}) + f_x(\boldsymbol{x}) \leq 0\}$ with $f_\xi(\boldsymbol{\xi}) = \gamma f_\xi^-(\boldsymbol{\xi}) - f_\xi^+(\boldsymbol{\xi})$ and $f_x(\boldsymbol{x}) = \gamma f_x^-(\boldsymbol{x}) - f_x^+(\boldsymbol{x})$. By assumption, $f_\xi(\boldsymbol{\xi})$ is an l.s.c. convex and $f_x(\boldsymbol{x})$ an l.s.c. increasing (decreasing) function. Further, $\mathcal{X}$ is normal (conormal) within a box and Ξ is convex. Observe that $\mathcal{D}$ is a convex set in $\boldsymbol{\xi}$ since for fixed $\boldsymbol{x}$, $f_x(\boldsymbol{x})$ is a constant, $\tilde{f}_\xi(\boldsymbol{\xi}) = f_\xi(\boldsymbol{\xi}) + \text{const.}$ is a convex function, and $\{\boldsymbol{\xi} : \tilde{f}_\xi(\boldsymbol{\xi}) \leq 0\}$ is a closed convex set [16, Thms. 4.6 & 7.1]. By the same argument, $\tilde{f}_x(\boldsymbol{x})$ is an increasing (decreasing) function and $\{\boldsymbol{x} : \tilde{f}_x(\boldsymbol{x}) \leq 0\}$ is a closed normal (conormal) set [11, Prop. 11.2]. Thus, $\mathcal{D}$ is normal (conormal) in a box in $\boldsymbol{x}$. Neither closed convex nor closed (co-)normal sets have any isolated feasible points [10]. Since $\mathcal{D}$ is either convex or (co-)normal in each coordinate the proposition is proven. □

3.1. A Branch and Bound Procedure to Solve (P4)

The BB procedure should only branch over the global variables $\boldsymbol{x}$. It successively partitions the $\boldsymbol{x}$-dimensions of $\mathcal{D}$ into boxes $\{\mathcal{M}_i\}$. For each box $\mathcal{M}_i$, a lower bound $\beta(\mathcal{M}_i)$ for (P4) with additional constraint $\boldsymbol{x} \in \mathcal{M}_i$ is computed. This bound relies on the following proposition.

Proposition 2 ([15, Proposition 2]). *Let $\bar{\boldsymbol{x}}^*_\mathcal{M}$ be a common maximizer of $\{g_i^-(\boldsymbol{x})\}_{i=1,\ldots,m}$ over the box $\mathcal{M}$. Then,* (P4)*'s objective is lower bounded over $\mathcal{M}$ by*

$$\max_{i=1,2,\ldots,m} \left\{ g_i^+(\boldsymbol{x}, \boldsymbol{\xi}) - g_i^-(\bar{\boldsymbol{x}}^*_\mathcal{M}) \right\} \tag{2}$$

This bound is tight at $\bar{\boldsymbol{x}}^*_\mathcal{M}$.

Thus, a lower bound for (P4) is the solution of

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{\xi}} & \max\limits_{i=1,2,\ldots,m} \left\{ g_{i,\xi}^+(\boldsymbol{\xi}) + g_{i,x}^+(\boldsymbol{x}) - g_i^-(\bar{\boldsymbol{x}}^*_{\mathcal{M}_i}) \right\} \\ \text{s.t.} & \gamma f_\xi^-(\boldsymbol{\xi}) - f_\xi^+(\boldsymbol{\xi}) + \gamma f_x^-(\boldsymbol{x}) - f_x^+(\boldsymbol{x}) \leq 0 \\ & \boldsymbol{\xi} \in \Xi,\ \boldsymbol{x} \in \mathcal{X} \cap \mathcal{M}_i. \end{cases} \tag{P5}$$

With the assumptions in Section 2 we can further simplify this problem. Let $\underline{\boldsymbol{x}}^*_{\mathcal{M}_i}$ be the common minimizer of $\gamma f_x^-(\boldsymbol{x}) - f_x^+(\boldsymbol{x})$, $g_{1,x}^+(\boldsymbol{x}), \ldots, g_{m,x}^+(\boldsymbol{x})$ over $\mathcal{X} \cap \mathcal{M}_i$. It is easy to see that $\underline{\boldsymbol{x}}^*_{\mathcal{M}_i}$ is the optimal solution of (P5) since it jointly minimizes the objective and the left-hand side of the constraint. Then, we obtain a simpler bound on (P4) by solving

$$\begin{cases} \min\limits_{\boldsymbol{\xi}} & \max\limits_{i=1,2,\ldots,m} \left\{ g_{i,\xi}^+(\boldsymbol{\xi}) + g_{i,x}^+(\underline{\boldsymbol{x}}^*_{\mathcal{M}_i}) - g_i^-(\bar{\boldsymbol{x}}^*_{\mathcal{M}_i}) \right\} \\ \text{s.t.} & \gamma f_\xi^-(\boldsymbol{\xi}) - f_\xi^+(\boldsymbol{\xi}) + \gamma f_x^-(\underline{\boldsymbol{x}}^*_{\mathcal{M}_i}) - f_x^+(\underline{\boldsymbol{x}}^*_{\mathcal{M}_i}) \leq 0 \\ & \boldsymbol{\xi} \in \Xi. \end{cases} \tag{P6}$$

This is, in general, a convex optimization problem and can be solved in polynomial time using standard tools [17]. For our motivating resource allocation problem (P3), this is easily verified to be a linear program since $g_{i,\xi}^+(\boldsymbol{\xi}) = \boldsymbol{a}_i^T$ and $\gamma f_\xi^-(\boldsymbol{\xi}) - f_\xi^+(\boldsymbol{\xi}) = -\sum_k \xi_k$.

In iteration k, the BB procedure selects the box $\mathcal{M}^k$ with the best bound, i.e., $\mathcal{M}^k = \arg\min_i \beta(\mathcal{M}_i)$. This box is then bisected adaptively via $(\boldsymbol{v}^k, j_k)$ where $\boldsymbol{v}^k = \frac{1}{2}(\boldsymbol{x}^k + \boldsymbol{y}^k)$ and $j_k = \arg\max_j \left| y_j^k - x_j^k \right|$, i.e., $\mathcal{M}^k$ is replaced by

$$\begin{aligned} \mathcal{M}_-^k &= \{\boldsymbol{x} \,|\, p_{j_k}^k \leq x_{j_k} \leq v_{j_k}^k, p_i^k \leq x_i \leq q_i^k \ (i \neq j_k)\} \\ \mathcal{M}_+^k &= \{\boldsymbol{x} \,|\, v_{j_k}^k \leq x_{j_k} \leq q_{j_k}^k, p_i^k \leq x_i \leq q_i^k \ (i \neq j_k)\}. \end{aligned} \tag{3}$$

The points $\boldsymbol{x}^k, \boldsymbol{y}^k \in \mathcal{M}^k$ in the computation of $\boldsymbol{v}^k$ need to satisfy

$$(\boldsymbol{x}^k, \boldsymbol{\xi}^k) \in \mathcal{D}, \quad \lim_{k\to\infty} \left(\beta(\mathcal{M}^k) - \min_{\boldsymbol{\xi} \in \mathcal{D}_{\boldsymbol{y}^k}} g(\boldsymbol{y}^k, \boldsymbol{\xi}) \right) = 0 \tag{4}$$

for convergence of this procedure [11, Prop. 6.2], where $\mathcal{D}_{\boldsymbol{y}^k}$ is $\mathcal{D}$ for fixed $\boldsymbol{x} = \boldsymbol{y}^k$ and $g(\boldsymbol{x}, \boldsymbol{\xi}) = \max_{i=1,2,\ldots,m} \left(g_i^+(\boldsymbol{x}, \boldsymbol{\xi}) - g_i^-(\boldsymbol{x}) \right)$. For $\beta(\mathcal{M}_i)$ being the solution of (P6) this is satisfied for $\beta(\mathcal{M}_i) = -\infty$ if (P6) is infeasible; and $\boldsymbol{y}^k = \bar{\boldsymbol{x}}^*_{\mathcal{M}^k}$ and $(\boldsymbol{x}^k, \boldsymbol{\xi}^k) = (\underline{\boldsymbol{x}}^*_{\mathcal{M}_i}, \boldsymbol{\xi}^*)$ otherwise, where $\boldsymbol{\xi}^*$ is the optimal solution of (P6). The following proposition connects the simple observation from Proposition 1 with the outlined BB procedure.

Proposition 3 (adapted from [11, Prop. 7.14]). *Let $\varepsilon > 0$ be given. Either $g(\boldsymbol{x}^k, \boldsymbol{\xi}^*) < 0$ for some k and $\boldsymbol{\xi}^*$ or $\beta(\mathcal{M}^k) > -\varepsilon$ for some k. In the former case, $(\boldsymbol{x}^k, \boldsymbol{\xi}^*)$ is a nonisolated feasible solution of* (P3) *satisfying $\frac{f^+(\boldsymbol{x}^k, \boldsymbol{\xi}^*)}{f^-(\boldsymbol{x}^k, \boldsymbol{\xi}^*)} \geq \gamma$. In the latter case, no ε-essential feasible solution $(\boldsymbol{x}, \boldsymbol{\xi})$ of* (P3) *exists such that $\frac{f^+(\boldsymbol{x}, \boldsymbol{\xi})}{f^-(\boldsymbol{x}, \boldsymbol{\xi})} \geq \gamma$.*

Thus, an adaptive BB algorithm for solving (P4) with deletion criterion $\beta(M) > -\varepsilon$ and stopping criterion $g(\boldsymbol{x}^k) < 0$ implements the feasibility check in the SIT scheme. The final procedure is stated in Algorithm 1 and its convergence is established below.

Theorem 1. *Algorithm 1 converges in finitely many steps to the (ε, η)-optimal solution of* (P3) *or establishes that no such solution exists.*

Proof. Convergence is mostly apparent from the discussion above (and [9–11]). It remains to show our choice of $\beta(\mathcal{M}_i)$ satisfies (4). Due to the branching procedure $\operatorname{diam} \mathcal{M}^k \to 0$ as $k \to \infty$. As $\operatorname{diam} \mathcal{M}^k$ shrinks, $\mathcal{M}^k \subseteq \mathcal{D}$ for some k. Also, $|\bar{\boldsymbol{x}}^*_{\mathcal{M}^k} - \underline{\boldsymbol{x}}^*_{\mathcal{M}^k}| \to 0$ and, thus, $\beta(\mathcal{M}^k) \to \min_{\boldsymbol{\xi} \in \mathcal{D}_{\boldsymbol{y}^k}} g(\boldsymbol{y}^k, \boldsymbol{\xi})$. Hence, (P6) satisfies (4) and Algorithm 1 converges. □

The initialization of Algorithm 1 requires a box $\mathcal{M}_0$ that contains the $\boldsymbol{x}$-dimensions of the feasible set. For (P1) this box is $[\boldsymbol{0}, \boldsymbol{P}]$. Observe that for the resource allocation problem (P1), problems (P7) and (P8) in Algorithm 1 are linear optimization problems.

4. NUMERICAL EVALUATION

We consider a Gaussian 3-user Gaussian multi-way relay channel [18, 19] with amplify-and-forward relaying and multiple unicast transmissions for the numerical evaluation of Algorithm 1 against the state-of-the-art. The achievable rate region with simultaneous non-unique decoding (SND) is stated in Proposition 4 below where the relay is node 0 and users are nodes 1 to 3, P_k, $\bar{P}_k$, and N_k are transmit, maximum transmit, and noise powers, respectively, $S_k = \frac{P_k}{N_0}$, h_k and g_k are up- and downstream channels, respectively, and $q(k)$ and $l(k)$ are index functions describing the message

Algorithm 1 SIT Algorithm for (P3)

0. Initialize $\varepsilon, \eta > 0$ and $\mathcal{M}_0 = [\boldsymbol{p}^0, \boldsymbol{q}^0]$, $\mathscr{P}_1 = \{\mathcal{M}_0\}$, $\mathscr{R} = \emptyset$, $k = 1$, and γ such that $\gamma \leq \frac{f^+(\boldsymbol{x},\boldsymbol{\xi})}{f^-(\boldsymbol{x},\boldsymbol{\xi})}$ for all feasible $(\boldsymbol{x}, \boldsymbol{\xi})$.

1. For each box $\mathcal{M} \in \mathscr{P}_k$:
- Set $\beta(\mathcal{M})$ as the solution of (P6) or $\beta(\mathcal{M}) = \infty$ if (P6) is infeasible.
- Add $\mathcal{M}$ to $\mathscr{R}$ if $\beta(\mathcal{M}) \leq -\varepsilon$.

2. Terminate if $\mathscr{R} = \emptyset$: If $\bar{\boldsymbol{x}}$ is not set, then (P3) is ε-essential infeasible; else $\bar{\boldsymbol{x}}$ is an essential (ε, η)-optimal solution.

3. Let $\mathcal{M}^k = \arg\min\{\beta(\mathcal{M}) \,|\, \mathcal{M} \in \mathscr{R}\}$ and solve the feasibility problem

$$\begin{cases} \text{find} & \boldsymbol{\xi} \in \Xi \\ \text{s.t.} & g_i^+(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k}, \boldsymbol{\xi}) - g_i^-(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k}) \leq 0, \; i = 1, \dots, m. \end{cases} \tag{P7}$$

If (P7) is feasible go to Step 4; otherwise go to Step 5.

4. $\underline{\boldsymbol{x}}^*_{\mathcal{M}^k}$ is a nonisolated feasible solution satisfying $\frac{f^+(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k},\boldsymbol{\xi})}{f^-(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k},\boldsymbol{\xi})} \geq \gamma$ for some $\boldsymbol{\xi} \in \Xi$. Solve

$$\begin{cases} \min\limits_{\boldsymbol{\xi} \in \Xi} & \dfrac{f^+(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k}, \boldsymbol{\xi})}{f^-(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k}, \boldsymbol{\xi})} \\ \text{s.t.} & g_i^+(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k}, \boldsymbol{\xi}) - g_i^-(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k}) \leq 0, \; i = 1, \dots, m. \end{cases} \tag{P8}$$

If $\bar{\boldsymbol{x}}$ is not set or $v(\text{P8}) > \gamma - \eta$, set $\bar{\boldsymbol{x}} = \underline{\boldsymbol{x}}^*_{\mathcal{M}^k}$ and $\gamma = v(\text{P8}) + \eta$.

5. Bisect $\mathcal{M}^k$ via $(\boldsymbol{v}^k, j_k)$ where $j_k \in \arg\max_j\{|\bar{x}^*_{\mathcal{M}^k,j} - \underline{x}^*_{\mathcal{M}^k,j}\}$ and $\boldsymbol{v}^k = \frac{1}{2}(\underline{\boldsymbol{x}}^*_{\mathcal{M}^k} + \bar{\boldsymbol{x}}^*_{\mathcal{M}^k})$ (cf. (3)). Remove $\mathcal{M}^k$ from $\mathscr{R}$. Let $\mathscr{P}_{k+1} = \{\mathcal{M}^k_-, \mathcal{M}^k_+\}$. Increment k and go to Step 1.

transfer. Please refer to [14, 20] for a detailed system model. Observe that this region is strictly larger than previously published SND regions [14, 20, 21] and includes treating interference as noise (IAN) as a special case. This is due to recent insights on SND decoders [22].

Proposition 4 ([15, Lem. 2])**.** *A rate triple (R_1, R_2, R_3) is achievable for the Gaussian MWRC with AF and SND if, for each $k \in \mathcal{K}$,*

$$R_k \leq \log\left(1 + \frac{|h_k|^2 S_k}{\gamma_k(\boldsymbol{S})}\right) \tag{5}$$

or

$$R_k \leq \log\left(1 + \frac{|h_k|^2 S_k}{\delta_k(\boldsymbol{S})}\right) \tag{6a}$$

$$R_k + R_{l(k)} \leq \log\left(1 + \frac{|h_k|^2 S_k + |h_{l(k)}|^2 S_{l(k)}}{\delta_k(\boldsymbol{S})}\right) \tag{6b}$$

where $S_k \leq \bar{S}_k$, $\delta_k(\boldsymbol{S}) = 1 + \widetilde{g}^{-1}_{q(k)}\left(1 + \sum_{i\in\mathcal{K}} |h_i|^2 S_i\right)$ with $\widetilde{g}_k = |g_k|^2 \frac{\bar{P}_0}{N_k}$, and $\gamma_k(\boldsymbol{S}) = \delta_k(\boldsymbol{S}) + |h_{l(k)}|^2 S_{l(k)}$.

Let $\mathcal{R}_{k,\text{IAN}}$ and $\mathcal{R}_{k,\text{SND}}$ be the regions defined by (5) and (6), respectively. Then, the rate region in Proposition 4 is $\mathcal{R} = \bigcap_{k\in\mathcal{K}} (\mathcal{R}_{k,\text{IAN}} \cup \mathcal{R}_{k,\text{SND}}) = \bigcup_{\boldsymbol{d}\in\{\text{IAN},\text{SND}\}^{|\mathcal{K}|}} \bigcap_{k\in\mathcal{K}} \mathcal{R}_{k,d_k}$. Since $\inf_{\boldsymbol{x}\in\bigcup_i \mathcal{D}_i} f(\boldsymbol{x}) = \min_i \inf_{\boldsymbol{x}\in\mathcal{D}_i} f(\boldsymbol{x})$, we can split the resource allocation problem into eight individual optimization problems. Each is easily identified as an instance of (P1) and solvable with Algorithm 1 using the initial box $\mathcal{M}_0 = [\boldsymbol{0}, \bar{\boldsymbol{S}}]$.

We assume equal maximum power constraints and noise power at all nodes. Channels are reciprocal and chosen independent and identically distributed (i.i.d.) as $\mathcal{CN}(0, 1)$. The static circuit power consumption $P_c = 1$ W, the power amplifier inefficiencies $\phi_i = 4$, $\varepsilon = 10^{-5}$, and the relay always transmits at maximum power. Results for SND, "traditional" SND defined by (6), and IAN defined by (5) are displayed in Fig. 1. First, observe that the curves saturate starting from 30 dB as is common for energy efficiency (EE) maximization. In this saturation region, all three approaches achieve the same EE. However, for lower signal-to-noise ratios (SNRs), IAN outperforms "traditional" SND. Of course, the EE performance depends quite a lot on the choice of real-world simulation parameters [23, 24], so further work is necessary to draw final conclusions in this regard.

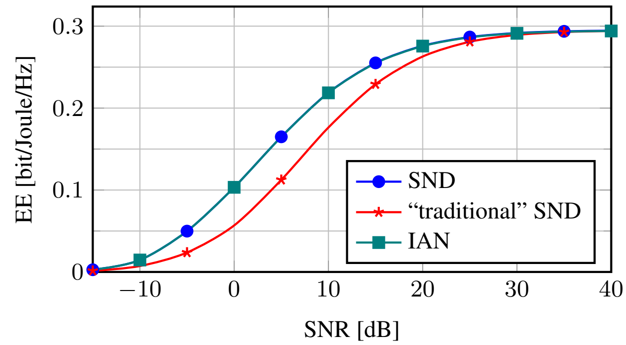

Fig. 1. Energy efficiency in the MWRC with AF relaying. Averaged over 1000 i.i.d. channel realizations and computed with $\eta = 10^{-3}$.

SNR		0 dB	20 dB	40 dB
Algorithm 1	Mean	5.1438 s	0.1771 s	0.155 s
	Median	3.2781 s	0.0762 s	0.06 s
Dinkelbach	Mean	377.1501 s	145.4181 s	36.969 s
	Median	162.811 s	23.027 s	16.9229 s

Table 1. Mean and median run times of EE computation for "traditional" SND and different solvers, all with precision $\eta = 0.01$

The main point of this section, however, is measuring the performance gain of inherently treating the fractional objective in Algorithm 1 over the state-of-the-art approach of using Dinkelbach's Algorithm [6–8].[1] Mean and median computation times for both approaches are reported in Table 1. Dinkelbach's Algorithm requires the global solution of a sequence of auxiliary problems that are solved with Algorithm 1. Hence, the differences in the run time are solely due to the use of Dinkelbach's Algorithm. It can be observed from Table 1 that our algorithm is always significantly faster (up to 800× on average at 20 dB) than Dinkelbach's Algorithm. Moreover, the obtained result is guaranteed to lie within an η-region around the true essential optimal value, which is not the case for Dinkelbach's Algorithm.

5. CONCLUSIONS

We introduce ε-essential feasibility and the accompanying SIT approach. Based on these concepts, we develop a novel global EE maximization algorithm that does not require Dinkelbach's iterative procedure and preserves the polynomial complexity in the number of convex variables. Numerical experiments show that our approach outperforms Dinkelbach's algorithm by almost four orders of magnitude.

[1] All algorithms are implemented in C++ with similar techniques. Thus, performance differences should be mostly due to algorithmic differences.

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