

ConCrete MAP: Learning a Probabilistic Relaxation of Discrete Variables for Soft Estimation with Low Complexity

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Abstract—Following the great success of Machine Learning (ML), especially Deep Neural Networks (DNNs), in many research domains in 2010s, several learning-based approaches were proposed for detection in large inverse linear problems, e.g., massive MIMO systems. The main motivation behind is that the complexity of Maximum A-Posteriori (MAP) detection grows exponentially with system dimensions. Instead of using DNNs, essentially being a black-box in its most basic form, we take a slightly different approach and introduce a probabilistic Continuous relaxation of disCrete variables to MAP detection. Enabling close approximation and continuous optimization, we derive an iterative detection algorithm: ConCrete MAP Detection (CMD). Furthermore, by extending CMD to the idea of deep unfolding, we allow for (online) optimization of a small number of parameters to different working points while limiting complexity. In contrast to recent DNN-based approaches, we select the optimization criterion and output of CMD based on information theory and are thus able to learn approximate probabilities of the individual optimal detector. This is crucial for soft decoding in today's communication systems. Numerical simulation results in MIMO systems reveal CMD to feature a promising performance complexity trade-off compared to SotA. Notably, we demonstrate CMD's soft outputs to be reliable for decoders.

Index Terms—Maximum a-posteriori (MAP), Individual optimal, Massive MIMO, Concrete distribution, Gumbel-softmax, Machine learning, Neural networks

I. INTRODUCTION

COMMUNICATIONS is a long standing engineering discipline whose theoretical foundation was laid by Claude Shannon with his landmark paper "A Mathematical Theory of Communication" in 1948 [1]. Since then, the theory has evolved into an own field known as information theory today and found its way into many other research areas where data or information is processed including artificial intelligence and especially its subdomain Machine Learning (ML). Information theory relies heavily on description with probabilistic models playing a significant role for design of new generations of cellular communication systems from 2-6G with respective increases in data rate. Probabilistic models have shown to be advantageous also in the ML research domain. Accordingly, both fields, communications and ML, have touched repeatedly in the past, e.g. [2], [3], [4].

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In the early 2010s, a special class of these models gave rise to several breakthroughs in data-driven ML research: Deep Neural Networks (DNNs). Inspired by the brain, several layers of artificial neurons are stacked on top of each other to create an expressive feed forward DNN able to approximate arbitrarily well [5] and thus to learn higher levels of abstraction, i.e., features, present in data [6]. This is of crucial importance for tasks where there are no well-established models but data to be collected. Previously considered intractable to optimize, dedicated hardware and software, i.e., Graphics Processing Units (GPU) and automatic differentiation frameworks [7], innovation to DNN models [8], [9] and advancements in training [8] have made it possible to build algorithms that equal or even surpass human performance in specific tasks such as pattern recognition [10] and playing games [11]. The impact included all ML subdomains, e.g., classification [9], [10] in supervised learning, generative modeling in unsupervised learning [12] and Q-learning in reinforcement learning [11].

A. ML in Communications

The great success of DNNs in many domains has stimulated large amount of work in communications just in recent years [6]. Especially in problems with a model deficit, e.g., detection in molecular and fiber-optical channels [13], [14], or without any known analytical solution, e.g., finding codes for AWGN-channels with feedback [15], DNNs have already proven to allow for promising application. Notably, the authors of the early work [16] demonstrate a complete communication system design by interpreting transmitter, channel and receiver as an autoencoder which is trained end-to-end similar to one DNN. The resulting encodings are shown to reach the BER performance of handcrafted systems in a simple AWGN scenario. A model-free approach based on reinforcement learning is proposed in [17]. Using advances in unsupervised learning, also blind channel equalization can be improved [18].

In contrast to typical ML research areas, a model deficit does not apply to wireless communications. The models, e.g., AWGN, describe reality well and enable development of optimized algorithms. However, this approach has its limits and the algorithms may be too complex to be implemented. This algorithm deficit applies to the core problem typical for communications: classification in large inverse problems. Therefore, it is crucial to find an approximate solution with an excellent trade-off between performance and complexity.

B. Related Work

A prominent example for large inverse problems under current deep investigation and a key enabler for better spectral efficiency in 5G/6G are massive Multiple Input Multiple Output (MIMO) systems [19]. In an uplink scenario, a Base Station (BS) is equipped with a very large number of antennas (around 64–256) and simultaneously serves multiple single-antenna User Equipments (UEs) on the same time-frequency resource. As a first step in receiver design, different tasks such as channel equalization/estimation and decoding are typically separated to lower complexity. But still, an algorithm deficit applies to both MIMO detection and decoding of large block-length codes, e.g., LDPC and Polar codes, since Maximum A-Posteriori (MAP) detection has high computational complexity growing exponentially with system or code dimensions. Even its efficient implementation, the Sphere Decoder (SD), remains too complex in such a scenario [20].

Hence, in communications history, many suboptimal solutions have been proposed to overcome the complexity bottleneck of the optimal detectors. One key approach is to relax the discrete Random Variables (RV) to be continuous: Remarkable examples include Matched Filter (MF), Zero Forcing and MMSE equalization. But linear equalization with subsequent detection shows bad performance especially in large symmetric systems.

A heuristic based on the latter is the V-Blast algorithm which first equalizes and then detects one layer with largest Signal-to-Noise Ratio (SNR) successively to reduce interference iteratively. A more efficient and sophisticated implementation, MMSE Ordered Successive Interference Cancellation (MOSIC), is based on a sorted QR decomposition of a MMSE extended system matrix with post sorting and offers a good trade-off between complexity and performance [21].

Pursuing another philosophy of mathematical optimization, the SemiDefinite Relaxation (SDR) technique [22] treats MIMO detection as a non-convex homogeneous quadratically constrained quadratic problem and relaxes it to be convex by dropping the only non-convex requirement. Proving to be a close approximation, SDR is more complex than MOSIC and solved by interior point methods from convex optimization.

Furthermore, also probabilistic model-based ML techniques were introduced to improve the trade-off and to integrate detection seamlessly with decoding: Mean Field Variational Inference (MFVI) provides a theoretical derivation of soft Successive Interference Cancellation (SIC) and the Bethe approach lays the foundation for loopy belief propagation [23]. Simplifying the latter, Approximate Message Passing (AMP) is derived known to be optimal for large system dimensions in i.i.d. Gaussian channels and computationally cheap [24]. As a further benefit, soft outputs are computed, today a strict requirement to account for subsequent soft decoding. But in practice, the performance of probabilistic approximations like MFVI and AMP suffers if the approximating conditions are not met, i.e., from the full-connected graph structure and finite dimensions in MIMO systems, respectively.

More recent work considers DNNs for application in MIMO systems and focus on the idea of deep unfolding [25], [26].

In deep unfolding, the number of iterations of a model-based iterative algorithm is fixed and its parameters untied. Further, it is enriched with additional weights and non-linearities to create a computational efficient DNN being optimized for performance improvements in MIMO detection [27], [28], belief propagation decoding [29], [30], [31] and MMSE channel estimation [32]. The former approach DetNet, a generic DNN model with a large number of trainable parameters based on an unfolded projected gradient descent, proves DNNs to allow for a promising trade-off between performance and complexity. In [33], unfolding of an extension of AMP to unitarily-invariant channels, the Orthogonal AMP (OAMP), into OAMPNet is proposed adding only 2 trainable parameters per layer. Offering promising performance, the complexity bottleneck of one matrix inversion per iteration makes this model-driven approach rather unattractive compared to DetNet. Another DNN-like network MMNet is inspired by iterative soft thresholding algorithms and AMP [34]: Striking the balance between expressiveness and complexity, and exploiting spectral and temporal locality, MMNet can be trained online for any realistic channel realization if coherence time is large enough. Since implementation of online training is not trivial, we focus in this work on the offline learned version MMNet (iid). One major drawback of the latter approaches is that they focus on MIMO detection and do not provide soft outputs.

C. Main Contributions

The main contributions of this article are manifold: Inspired by recent ML research, we first introduce a CONtinuous relaxation of the probability mass function (pmf) of the disCRETE RVs by a probability density function (pdf) from [35], [36] to the MAP detection problem. The proposed CONCRETE relaxation offers many favorable properties: On the one hand, the pdf of continuous RVs converges to the exact pmf in the parameter limit. On the other hand, we notice good algorithmic properties like avoiding marginalization and allowing for differentiation instead. By this means, we replace exhaustive search by computationally cheaper continuous optimization to approximately solve the MAP problem in any probabilistic non-linear model. We name our approach Concrete MAP Detection (CMD).

Second, following the idea of Deep Unfolding, we unfold the gradient descent algorithm into a DNN-like model with a fixed number of iterations to allow for parameter optimization and to further improve detection performance while limiting detection complexity. By this means, we are able to combine the advantages of DNNs and model-based approaches. As the number of parameters is small, we are able to dynamically adapt them to easily adjust CMD to different working points. Further, the resulting structure allows for fast online training of CMD.

Thirdly, we derive the optimization criterion from an information theoretic perspective and are hence able to provide probabilities of detection, i.e., reliable soft outputs. We show that optimization is then equivalent to learning an approximation of the Individual Optimal (IO) detector. This allows us to account for subsequent decoding, e.g., in MIMO systems, in contrast to literature [28], [34].

Finally, we provide numerical simulation results for use of CMD in MIMO systems including a variety of simulation setups, e.g., correlated channels, revealing CMD to be a generic and promising approach competitive to State of the Art (SotA). Notably, we show superiority to other recently proposed ML-based approaches and demonstrate with simulations in coded systems CMD's soft outputs to be reliable for decoders as opposed to [28]. Furthermore, by estimating the computational complexity, we prove CMD to feature an excellent trade-off between performance and complexity. Notably, only the Matched Filter has lower complexity.

In the following, we first introduce the concrete relaxation to MAP detection in Section II using the example of an inverse linear problem. In Section III, we follow a different route and explain how to learn the posterior, i.e., replacing it by some tractable approximation. To yield a suitable model for this approximation, we propose to unfold CMD which we are then able to train by variants of Stochastic Gradient Descent (SGD). Finally in Section IV and V, we provide numerical results of the bit error performance in comparison to other SotA approaches using the example of uncoded and coded MIMO systems and summarize the main results, respectively.

II. CONCRETE RELAXATION OF MAP PROBLEM

A. System Model and Problem Statement

To motivate the concrete relaxation, we consider a linear complex-valued observation model typically encountered in communications, e.g., MIMO systems, first excluding coding:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}. \quad (1)$$

Here, we assume \mathbf{x} to be a normalized multivariate discrete RV, i.e., $\mathbf{x} = \{x_n\}_{n=1}^{N_T}$ with $E[|x_n|^2] = 1$, whose i.i.d. elements are from a set \mathcal{M} , e.g., 16-QAM or 8-PSK. A linear channel $\mathbf{H} \in \mathbb{C}^{N_R \times N_T}$ with i.i.d. Gaussian taps $h_{mn} \sim \mathcal{CN}(0, 1/N_R)$ introduces correlation. Then, the resulting RV is superimposed by Gaussian noise $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2 \mathbf{I}_{N_R})$ with variance σ_n^2 . The matrix \mathbf{I}_{N_R} denotes the identity matrix of dimension $N_R \times N_R$.

To detect the discrete multivariate RV \mathbf{x} given the linear observation $\mathbf{y} \in \mathbb{C}^{N_R \times 1}$, there exist two optimal detectors from a probabilistic Bayesian viewpoint: First, we can use Bayes rule to formulate the MAP problem

$$\hat{\mathbf{x}} = \arg \max_{\mathbf{x} \in \mathcal{M}^{N_T \times 1}} p(\mathbf{x}|\mathbf{y}) \quad (2a)$$

$$= \arg \max_{\mathbf{x} \in \mathcal{M}^{N_T \times 1}} p(\mathbf{y}|\mathbf{x}) \cdot p(\mathbf{x}) \quad (2b)$$

$$= \arg \min_{\mathbf{x} \in \mathcal{M}^{N_T \times 1}} -\ln p(\mathbf{y}|\mathbf{x}) - \ln p(\mathbf{x}) \quad (2c)$$

$$\text{with } p(\mathbf{y}|\mathbf{x}) = \frac{1}{\pi^{N_R} \sigma_n^{2N_R}} e^{-\frac{1}{\sigma_n^2} (\mathbf{y}-\mathbf{H}\mathbf{x})^H (\mathbf{y}-\mathbf{H}\mathbf{x})} \quad (3)$$

as the likelihood function and $p(\mathbf{x})$ as the a-priori pdf. Since the RV is discrete, i.e., $x_n \in \mathcal{M}$, an exhaustive search over all element combinations is required to solve the MAP problem becoming computational intractable for large system dimensions. Note that the Sphere Detector (SD) provides an efficient implementation [20]. Second, we notice that the MAP detector only delivers the most likely received vector

\mathbf{x} and hence hard decisions. In coded systems with soft decoders usually employed today, delivering soft information is a strict requirement. This brings us to the Individual Optimal (IO) detector obtained by evaluating the marginal posterior distribution w.r.t. every single x_n :

$$\hat{x}_n = \arg \max_{x_n \in \mathcal{M}} p(x_n|\mathbf{y}) = \arg \max_{x_n \in \mathcal{M}} \frac{\sum_{\mathbf{x} \setminus x_n} p(\mathbf{y}|\mathbf{x}) \cdot p(\mathbf{x})}{\sum_{x_n} \sum_{\mathbf{x} \setminus x_n} p(\mathbf{y}|\mathbf{x}) \cdot p(\mathbf{x})}. \quad (4)$$

This detector is optimal in terms of minimizing the Symbol Error Rate (SER) per individual symbol without coding and further delivers probabilities as soft output in contrast to MAP detection. However, it has higher complexity due to required marginalization w.r.t. \mathbf{x} . Since the MAP detector performance coincides with the IO detector in the high SNR regime and is of lower complexity, we restrict to the MAP detector as a benchmark in simulations without coding.

B. Concrete Distribution

We now focus on the following question to improve the performance complexity trade-off: How to model the prior information $p(\mathbf{x})$ accurately by some approximation $p(\tilde{\mathbf{x}})$? In [37], we proposed to use ML tricks from [35], [36] to achieve this and to make inference computationally tractable. The idea was recently discovered in the ML community in the context of unsupervised learning of generative models [35], [36]. There, marginalization to compute the objective function, the evidence, becomes intractable. Therefore, the Evidence is replaced by its Lower Bound (ELBO) by means of an auxiliary posterior function. But optimizing w.r.t. the ELBO results in high variance of the gradient estimators. For variance reduction, the so called reparametrization trick is used and leads to an optimization structure similar to an auto-encoder known as the variational autoencoder [23]. There, the stochastic node is reparametrized by a continuous stochastic variable, e.g., a Gaussian, and its parameters, e.g., mean and variance. In contrast to continuous variables, reparametrization of discrete RV is not possible. Hence, a Continuous relaxation of discrete variables, the CONCRETE distribution, was proposed in [35], [36] independently.

To explain the introduction of this relaxation to the MAP problem, let us assume that we have the discrete binary RV $x \in \mathcal{M}$ with $\mathcal{M} = \{-1, +1\}$. Further, we define the discrete RV \mathbf{z} as a one-hot vector where all elements are zero except for one element, i.e., $\mathbf{z} \in \{0, 1\}^{2 \times 1}$ with two possible realizations $\mathbf{z}_1 = [1, 0]^T$, $\mathbf{z}_2 = [0, 1]^T$. In addition, we describe the values of \mathcal{M} by the representer vector $\mathbf{m} = [-1, 1]^T$. That way, we can write $x = \mathbf{z}^T \mathbf{m}$, e.g., $x = [1, 0] \cdot [-1, 1]^T = -1$. Now, the one-hot vector $\mathbf{z} \in \{0, 1\}^{M \times 1}$ represents a categorical RV with $M = |\mathcal{M}|$ classes. Connecting Monte Carlo methods to optimization [35], the Gumbel-Max trick states that we are able to generate samples, i.e., classes, of such a categorical RV or pmf $p(x)$ by sampling an index i^* from M continuous i.i.d. Gumbel RVs g_i known from extreme value theory:

$$i^* = \arg \max_{i=1, \dots, M} \ln p(x = m_i) + g_i. \quad (5)$$

Defining the function $\text{one-hot}(i^*)$ which sets the i^* -th element in the one-hot vector $z_{i^*} = 1$ and $z_{l \neq i^*} = 0$, the Gumbel-Max trick hence allows to sample one-hot vectors \mathbf{z} . Thus, we are able to reparametrize \mathbf{z} through a continuous multivariate Gumbel RV $\mathbf{g} \in \mathbb{R}^{M \times 1}$ and a vector $\boldsymbol{\alpha} \in [0, 1]^{M \times 1}$ of class probabilities $p(x = m_k)$ with $\sum_{k=1}^M \alpha_k = 1$:

$$\mathbf{z} = \text{one-hot} \left(\arg \max_{i=1, \dots, M} [\ln(\boldsymbol{\alpha}) + \mathbf{g}] \right). \quad (6)$$

Note that (6) and equally x are still discrete RVs, i.e., $p(\mathbf{z}) \hat{=} p(x)$, but represented in probabilistic sense by continuous RVs \mathbf{g} . To achieve a continuous RV we now replace the one-hot and arg max computation in (6) by the softmax function [35], [36]:

$$\tilde{\mathbf{z}} = \sigma_{\tau}(\mathbf{g}) = \frac{e^{(\ln(\boldsymbol{\alpha}) + \mathbf{g})/\tau}}{\sum_{i=1}^M e^{(\ln \alpha_i + g_i)/\tau}}. \quad (7)$$

The resulting RV $\tilde{\mathbf{z}} \in [0, 1]^{M \times 1}$ is the so called concrete or Gumbel-Softmax RV and now continuous, e.g., $\tilde{\mathbf{z}} = [0.2, 0.8]^T$. It is controlled by a parameter, the softmax temperature τ . The distribution of $\tilde{\mathbf{z}}$ in (7) was found to have a closed form density which gives the definition of the concrete distribution:

$$p(\tilde{\mathbf{z}}|\boldsymbol{\alpha}, \tau) = (M-1)! \tau^{M-1} \prod_{k=1}^M \left(\frac{\alpha_k \tilde{z}_k^{-\tau-1}}{\sum_{i=1}^M \alpha_i \tilde{z}_i^{-\tau}} \right). \quad (8)$$

With $\tilde{\mathbf{z}}$, we are finally able to relax the discrete RV x into a continuous RV \tilde{x} by defining $\tilde{x} = \tilde{\mathbf{z}}^T \mathbf{m}$. Now, our derivation of the relaxation is complete. In Fig. 1, we illustrate the distribution $p(\tilde{x})$ for the special case $M = 2$ in comparison to the original categorical pdf $p(x)$, i.e., a Bernoulli pmf. It has the following properties reflecting the correctness of the relaxation [35]: First, we are able to reparametrize the concrete RV $\tilde{\mathbf{z}}$ and hence the RVs \tilde{x} by Gumbel variables \mathbf{g} , a direct result from the initial idea (7). Moreover, the zero temperature limit $\tau \rightarrow 0$ restores a categorical variable: The smaller τ , the more $\tilde{\mathbf{z}}$ approaches a categorical distribution and the approximation becomes more accurate. Thus, the statistics of x and \tilde{x} remain the same for $\tau \rightarrow 0$.

C. Reparametrization

In [37], the idea is to use the concrete distribution in order to relax the MAP problem (2c) to

$$\hat{\mathbf{x}} = \arg \min_{\tilde{\mathbf{x}} \in [x_{\min}, x_{\max}]^{N_T \times 1}} -\ln p(\mathbf{y}|\tilde{\mathbf{x}}) - \ln p(\tilde{\mathbf{x}}). \quad (9)$$

The reparametrization of $\tilde{\mathbf{z}}$ by \mathbf{g} helps to rewrite (9) by expressing each \tilde{x}_n in $\tilde{\mathbf{x}}$ with (7) by the RV \mathbf{g}_n , $n = 1, \dots, N_T$ of i.i.d. Gumbel RVs g_{kn} :

$$\tilde{\mathbf{x}}(\mathbf{G}) = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{N_T} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{z}}_1^T \\ \vdots \\ \tilde{\mathbf{z}}_{N_T}^T \end{bmatrix} \mathbf{m} = \begin{bmatrix} \sigma_{\tau}(\mathbf{g}_1)^T \\ \vdots \\ \sigma_{\tau}(\mathbf{g}_{N_T})^T \end{bmatrix} \mathbf{m} \quad (10)$$

$$\text{with } \mathbf{G} = [\mathbf{g}_1 \ \dots \ \mathbf{g}_{N_T}] \in \mathbb{R}^{M \times N_T}. \quad (11)$$

By doing so, we will obtain an unconstrained optimization problem w.r.t. matrix \mathbf{G} . Now, we reformulate the relaxed

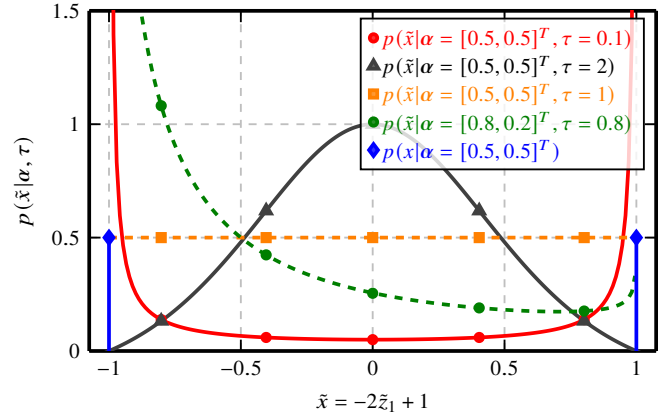


Fig. 1. The concrete pdf $p(\tilde{x}|\boldsymbol{\alpha}, \tau)$ shown for different parameter sets and $M = 2$. It relaxes the Bernoulli pmf $p(x|\boldsymbol{\alpha})$ into the interior. Notably, for $\tau \leq (M-1)^{-1}$, it is log-convex and log-concave otherwise. Symmetry results if $\alpha_1 = \dots = \alpha_M$.

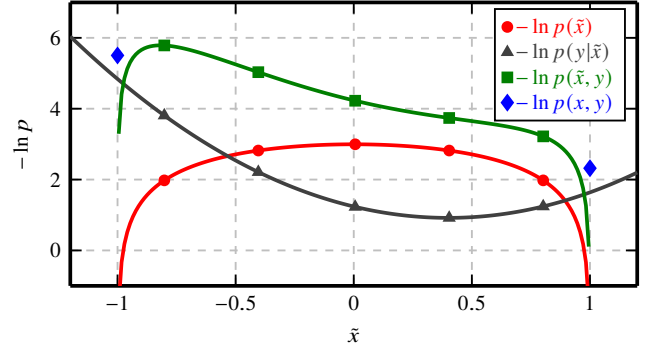


Fig. 2. Exemplary plot of the concrete binary MAP cost function (green) and the contribution of conditional (black) and prior pdf (red) to it. The original binary MAP cost function (blue) is shown for comparison.

MAP problem (9): This means, we replace the Gaussian model $p(\mathbf{y}|\tilde{\mathbf{x}})$ by $p(\mathbf{y}|\mathbf{G})$ and introduce the Gumbel distribution $p(g_{kn}) = \exp(-g_{kn} - \exp(-g_{kn}))$ as the new prior distribution:

$$\hat{\mathbf{G}} = \arg \min_{\mathbf{G} \in \mathbb{R}^{M \times N_T}} -\ln p(\mathbf{y}|\mathbf{G}) - \ln p(\mathbf{G}) \quad (12a)$$

$$= \arg \min_{\mathbf{G} \in \mathbb{R}^{M \times N_T}} -\ln p(\mathbf{y}|\mathbf{G}) - \sum_{n=1}^{N_T} \sum_{k=1}^M \ln p(g_{kn}) \quad (12b)$$

$$= \arg \min_{\mathbf{G} \in \mathbb{R}^{M \times N_T}} \frac{1}{\sigma_n^2} (\mathbf{y} - \mathbf{H}\tilde{\mathbf{x}}(\mathbf{G}))^H (\mathbf{y} - \mathbf{H}\tilde{\mathbf{x}}(\mathbf{G})) + N_R \ln(\pi \sigma_n^2) + \mathbf{1}^T \mathbf{G} \mathbf{1} + \mathbf{1}^T e^{-\mathbf{G}} \mathbf{1} \quad (12c)$$

$$= \arg \min_{\mathbf{G} \in \mathbb{R}^{M \times N_T}} L(\mathbf{G}, \tau). \quad (12d)$$

However, owing to the softmax and exponential terms in $L(\mathbf{G}, \tau)$, (12d) has no analytical solution. Furthermore, $L(\mathbf{G}, \tau)$ describes a non-convex objective function which is illustrated in Fig. 2 for the binary case $M = 2$. This results from log-convexity of the concrete distribution for $\tau \leq (M-1)^{-1}$ [35]. The conditional pdf $p(\mathbf{y}|\tilde{\mathbf{x}})$ is log-concave and the prior pdf $p(\tilde{\mathbf{x}})$ log-convex, so the negative log joint distribution forms a non-convex objective function (12d).

D. Gradient Descent Optimization

One common strategy for solving the non-linear and non-analytical problem (12d) is to use a variant of gradient descent based approaches. Since we aim to reduce complexity, we choose the most basic form steepest descent. The minimum is approached iteratively by taking gradient descent steps until the necessary condition

$$\frac{\partial L(\mathbf{G}, \tau)}{\partial \mathbf{G}} = \mathbf{0} \quad (13)$$

is fulfilled. We point out that convergence to the global solution depends heavily on the starting point initialization since the objective function is non-convex. A reasonable choice of starting point value is $\tilde{\mathbf{x}}^{(0)} = \mathbf{E}[\mathbf{x}] = \boldsymbol{\alpha}^T \cdot \mathbf{m}$, i.e., the expected value of the true discrete RV \mathbf{x} . We achieve this by setting $\mathbf{G}^{(0)} = \mathbf{0}$ and $\tau = 1$. After some tensor/matrix calculus and by noting that every \tilde{x}_n only depends on one \mathbf{g}_n , the gradient descent step for (12d) in iteration j is:

$$\mathbf{G}^{(j+1)} = \mathbf{G}^{(j)} - \delta^{(j)} \cdot \left. \frac{\partial L(\mathbf{G}, \tau)}{\partial \mathbf{G}} \right|_{\mathbf{G}=\mathbf{G}^{(j)}} \quad (14a)$$

$$\begin{aligned} \frac{\partial L(\mathbf{G}, \tau)}{\partial \mathbf{G}} &= \frac{2}{\sigma_n^2} \cdot \left[\frac{\partial \tilde{x}_1(\mathbf{g}_1)}{\partial \mathbf{g}_1} \quad \dots \quad \frac{\partial \tilde{x}_{N_T}(\mathbf{g}_{N_T})}{\partial \mathbf{g}_{N_T}} \right] \\ &\cdot \text{diag} \{ \mathbf{H}^H \mathbf{H} \tilde{\mathbf{x}}(\mathbf{G}) - \mathbf{H}^H \mathbf{y} \} \\ &+ 1 - e^{-\mathbf{G}} \end{aligned} \quad (14b)$$

$$\frac{\partial \tilde{x}_n(\mathbf{g}_n)}{\partial \mathbf{g}_n} = \frac{1}{\tau^{(j)}} \cdot [\text{diag} \{ \sigma_\tau(\mathbf{g}_n) \} \cdot \mathbf{m} - \sigma_\tau(\mathbf{g}_n) \cdot \tilde{x}_n(\mathbf{g}_n)] \quad (14c)$$

The operator $\text{diag} \{ \mathbf{a} \}$ creates a diagonal matrix with the vector \mathbf{a} on its main diagonal. The step-size $\delta^{(j)}$ can be chosen adaptively in every iteration j just as the parameter $\tau^{(j)}$. For example, we can follow a heuristic schedule like in simulated annealing: We start with a large $\tau^{(j)}$ and decrease until we approach the true prior pdf for $\tau^{(j)} \rightarrow 0$. Finally, after the last iteration N_{it} , we get as a result the continuous estimate $\mathbf{G}^{(N_{it})}$. For approximate detection of \mathbf{x} in (2c), the estimate has to be transformed back to the discrete domain by quantizing $\tilde{\mathbf{x}}$ onto the discrete set \mathcal{M} :

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathcal{M}^{N_T \times 1}} \left\| \mathbf{x} - \tilde{\mathbf{x}} \left(\mathbf{G}^{(N_{it})} \right) \right\|_2 \quad (15)$$

In the following, we name this detection approach Concrete MAP Detection (CMD). It is a generic approach applicable in any differentiable probabilistic non-linear model. Furthermore, only elementwise nonlinearities and matrix vector multiplications are present.

As a final remark, we note that our implementation of Section IV relies on scaling of the objective function by the noise variance parameter, i.e., $\sigma_n^2 \cdot L(\mathbf{G}, \tau)$. Although scaling does not change the optimization problem, we observed that this slightly modified version of (14) is numerically more stable.

E. Special Case: Binary Random Variables

Noting that the softmax function (7) is normalized, we are able to eliminate one degree of freedom in matrix $\mathbf{G} \in \mathbb{R}^{M \times N_T}$ along dimension M . For the special case of binary RVs or

$M = 2$ classes, this means that the matrix \mathbf{G} can be reduced to a vector $\mathbf{s} \in \mathbb{R}^{N_T \times 1}$ of logistic RVs to derive a different algorithm of low complexity. Here, we only briefly summarize the result of binary CMD in a real-valued system model and refer the reader to [37] for the complete derivation:

$$\mathbf{s}^{(j+1)} = \mathbf{s}^{(j)} - \delta^{(j)} \cdot \left. \frac{\partial L(\mathbf{s})}{\partial \mathbf{s}} \right|_{\mathbf{s}=\mathbf{s}^{(j)}} \quad (16a)$$

$$\frac{\partial L(\mathbf{s})}{\partial \mathbf{s}} = \frac{1}{\sigma_n^2} \cdot \frac{\partial \tilde{\mathbf{x}}(\mathbf{s})}{\partial \mathbf{s}} \cdot [\mathbf{H}^T \mathbf{H} \tilde{\mathbf{x}}(\mathbf{s}) - \mathbf{H}^T \mathbf{y}] + \tanh \left(\frac{\mathbf{s}}{2} \right) \quad (16b)$$

$$\frac{\partial \tilde{\mathbf{x}}(\mathbf{s})}{\partial \mathbf{s}} = \frac{1}{2\tau^{(j)}} \cdot \text{diag} \{ 1 - \tilde{\mathbf{x}}^2(\mathbf{s}) \} \quad (16c)$$

$$\tilde{\mathbf{x}}(\mathbf{s}) = \tanh \left(\frac{\ln(1/\alpha - 1) + \mathbf{s}}{2\tau} \right) \quad (16d)$$

The final step consists again of quantization - in this case it simplifies to the sign function: $\hat{\mathbf{x}} = \text{sign}(\tilde{\mathbf{x}}(\mathbf{s}^{(N_{it})}))$.

III. LEARNING TO RELAX

Although being simple and computational efficient, using a gradient descent approach like (14) and (16) leads to several inconveniences. Regarding theoretical properties, a major drawback becomes apparent: Convergence of the gradient descent steps to an optimum is slow since consecutive gradients are perpendicular. Also practical questions arise: How to choose the parameters $\tau^{(j)}$ and $\delta^{(j)}$ and the number of iterations N_{it} for a good complexity performance trade off? And how are we able to deliver soft information, e.g., probabilities, to a soft decoder which is standard in today's communication systems?

Our idea is to improve CMD by learning and in particular the idea of deep unfolding to address these questions. This means we have to deal with

- A. how learning is defined
- B. the application of deep unfolding to CMD.

A. Basic Problem of Learning

To introduce our notation of learning, we revisit our basic task of MAP detection. Ideally, we would like to infer the most likely transmit signal \mathbf{x} based on an a-posteriori pdf $p(\mathbf{x}|\mathbf{y})$. But as pointed out earlier, evaluation of $p(\mathbf{x}|\mathbf{y})$ has intractable complexity. For this reason, we propose to relax the MAP problem and CMD, respectively.

Another idea to tackle this problem is to approximate this pdf $p(\mathbf{x}|\mathbf{y})$ by another computationally tractable pdf $q(\mathbf{x}|\mathbf{y})$, e.g., by calculation of $q(\mathbf{x}|\mathbf{y})$ using few samples/observations \mathbf{x} , and use this pdf for inference. Note that this approach includes cases where we do not know the pdf $p(\mathbf{x}|\mathbf{y})$ completely. The quality of the approximation can be quantified by the information theoretic measure of Kullback-Leibler (KL) divergence:

$$D_{\text{KL}}(p \parallel q) = - \sum_{\mathbf{x} \in \mathcal{M}^{N_T \times 1}} p(\mathbf{x}|\mathbf{y}) \ln \frac{p(\mathbf{x}|\mathbf{y})}{q(\mathbf{x}|\mathbf{y})} \quad (17)$$

$$= \mathbf{E}_{\mathbf{x} \sim p(\mathbf{x}|\mathbf{y})} \left[\ln \frac{p(\mathbf{x}|\mathbf{y})}{q(\mathbf{x}|\mathbf{y})} \right] \quad (18)$$

Just as the Mean Square Error (MSE), the measure of KL divergence can be used to define an optimization problem targeting at a tight $q(\mathbf{x}|\mathbf{y})$ as a solution. This brings me to a crucial viewpoint of this article: **Learning is defined to be the optimization process aiming to derive a good approximation $q(\mathbf{x}|\mathbf{y})$ of $p(\mathbf{x}|\mathbf{y})$, i.e.,**

$$q^*(\mathbf{x}|\mathbf{y}) = \arg \min_q D_{\text{KL}}(p \parallel q). \quad (19)$$

This kind of problem is also referred to as Variational Inference (VI). We can rewrite the KL divergence into a sum of cross entropy $\mathcal{H}(p, q)$ and entropy $\mathcal{H}(p)$:

$$D_{\text{KL}}(p \parallel q) = \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x}|\mathbf{y})}[-\ln q(\mathbf{x}|\mathbf{y})] - \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x}|\mathbf{y})}[-\ln p(\mathbf{x}|\mathbf{y})] \quad (20)$$

$$= \mathcal{H}(p, q) - \mathcal{H}(p). \quad (21)$$

Since we defined the basic learning problem (19) w.r.t. approximation q , we can neglect the entropy term $\mathcal{H}(p)$ independent of q and use cross entropy as the learning criterion. If we further restrict q to a model $q(\mathbf{x}|\mathbf{y}, \theta)$ with parameters θ , the optimization problem now reads:

$$\theta^* = \arg \min_{\theta} \mathcal{H}(p, q). \quad (22)$$

We note that problem (22) is solved separately for each \mathbf{y} and thus not computationally efficient. Therefore, we define one inference distribution $q(\mathbf{x}|\mathbf{y}, \theta)$ for any value \mathbf{y} which is known as Amortized Inference [23]:

$$\theta^* = \arg \min_{\theta} \mathbb{E}_{\mathbf{y} \sim p(\mathbf{y})} [\mathcal{H}(p(\mathbf{x}|\mathbf{y}), q(\mathbf{x}|\mathbf{y}, \theta))] \quad (23)$$

$$= \arg \min_{\theta} \mathbb{E}_{\mathbf{y} \sim p(\mathbf{y})} [\mathbb{E}_{\mathbf{x} \sim p(\mathbf{x}|\mathbf{y})} [-\ln q(\mathbf{x}|\mathbf{y}, \theta)]] \quad (24)$$

$$\approx \arg \min_{\theta} -\frac{1}{N} \sum_{i=1}^N \ln q(\mathbf{x}_i|\mathbf{y}_i, \theta), \quad N \rightarrow \infty. \quad (25)$$

The final result (25) equals the maximum likelihood problem in supervised learning. We make use of it in the following since it allows for numerical optimization based on data points $\{\mathbf{x}_i, \mathbf{y}_i\}$. Furthermore, it proves to be a Monte Carlo approximation of (23) and is hence well motivated from information theory [23].

B. Idea of Unfolding and Application to CMD

Learning gives us the ability to obtain a tractable approximation $q(\mathbf{x}|\mathbf{y}, \theta)$. But it remains one question: How to choose a suitable functional form for $q(\mathbf{x}|\mathbf{y}, \theta)$ of low complexity and good performance? We follow the idea of deep unfolding from [26] and apply it to our model-based approach CMD with parameters $\theta = \{\tau^{(0)}, \dots, \tau^{(N_{\text{it}})}, \delta^{(0)}, \dots, \delta^{(N_{\text{it}}-1)}\} \in \mathbb{R}^{(2N_{\text{it}}+1) \times 1}$ able to relax tightly. Thereby, we combine strengths of DNNs and the latter: DNNs are known to be universal approximators [5] and their fixed structure of parallel computations layer per layer allows to define a good performance complexity trade off at run time. But if the model is dynamic and changes, e.g., the channel or noise over time, reiterated optimization of (22) is required and the benefit disappears. Fortunately, we know our model (3), a MIMO channel, well and are able to use generative model-based approaches which mostly rely on

a suitable approximation of (19) for computational tractability. For example, MFVI and AMP belong to this algorithm family.

This means we unfold the iterations (14) of CMD into a DNN by untying the parameters $\tau^{(j)}$ and $\delta^{(j)}$. Furthermore, we fix the complexity by setting the number of iterations N_{it} . The resulting graph illustrated in Fig. 3 for binary CMD has a DNN-like structure which should be able to generalize and approximate well at the same time. Owing to the skip connection from $\mathbf{s}^{(j)}$ to $\mathbf{s}^{(j+1)}$ on the right hand side, the structure resembles a Residual Network (ResNet) layer which is SotA in image processing [9]. It is a result of the gradient descent approach which allows to interpret optimization of ResNets as learning gradient descent steps. The reason for the success of ResNet lies in the skip connection: The training error is able to backpropagate through it to early layers which allows for fast adaptation of early weights and hence fast training of DNNs. This makes CMD especially suitable for online training proposed in [34] and allows for refinement in application.

As before, we have to define a final layer which is now also used for optimization. Usually, its output is chosen to be an continuous estimate of \mathbf{x} and optimized w.r.t. the MSE criterion, see [28], [34]. This viewpoint relaxes the estimate $\hat{\mathbf{x}}$ into $\mathbb{R}^{N_{\text{T}} \times 1}$ and assumes a Gaussian distribution for errors at the output. In our case, the output would correspond to $\tilde{\mathbf{x}}(\mathbf{G}^{(N_{\text{it}})})$ from (15). But this is in contrast to our information theoretic viewpoint on learning which states that we want to approximate an output of the true pmf $p(\mathbf{x}|\mathbf{y})$. Like in MFVI, we assume a factorization of the approximating posterior to make it computationally tractable and derive our learning criterion:

$$\mathcal{H}(p, q) = - \sum_{\mathbf{x} \in \mathcal{M}^{N_{\text{T}}}} p(\mathbf{x}|\mathbf{y}) \cdot \ln q(\mathbf{x}|\mathbf{y}, \theta) \quad (26)$$

$$\stackrel{\text{MFVI}}{=} - \sum_{\mathbf{x} \in \mathcal{M}^{N_{\text{T}}}} p(\mathbf{x}|\mathbf{y}) \cdot \ln \prod_{x_n \in \mathcal{M}} q_n(x_n|\mathbf{y}, \theta) \quad (27)$$

$$= - \sum_{x_n \in \mathcal{M}} \ln q_n(x_n|\mathbf{y}, \theta) \cdot \sum_{x_n \in \mathcal{M}^{N_{\text{T}}-1}} p(\mathbf{x}|\mathbf{y}) \quad (28)$$

$$= - \sum_{x_n \in \mathcal{M}} p(x_n|\mathbf{y}) \cdot \ln q_n(x_n|\mathbf{y}, \theta) \quad (29)$$

$$= \sum_{x_n \in \mathcal{M}} \mathcal{H}(p(x_n|\mathbf{y}), q_n(x_n|\mathbf{y}, \theta)). \quad (30)$$

This interesting result shows that assuming MFVI factorization leads to an optimization criterion w.r.t. the soft output $p(x_n|\mathbf{y})$ of the IO detector (4). This soft output is required for subsequent decoding and thus exactly what we need.

The last step of our idea consists of inserting our unfolded CMD structure into $q_n(x_n|\mathbf{y}, \theta)$. Hence, we propose to use a softmax function for the last layer being a typical choice for classification in discriminative probabilistic models. Fortunately, CMD already includes this softmax function as part of its structure so we rewrite

$$q_n(x_n|\mathbf{y}, \theta) = \prod_{k=1}^M q_{n,k}(x_n|\mathbf{y}, \theta)^{(x_n=m_k)} = \prod_{k=1}^M \tilde{z}_{n,k}^{(x_n=m_k)} \quad (31)$$

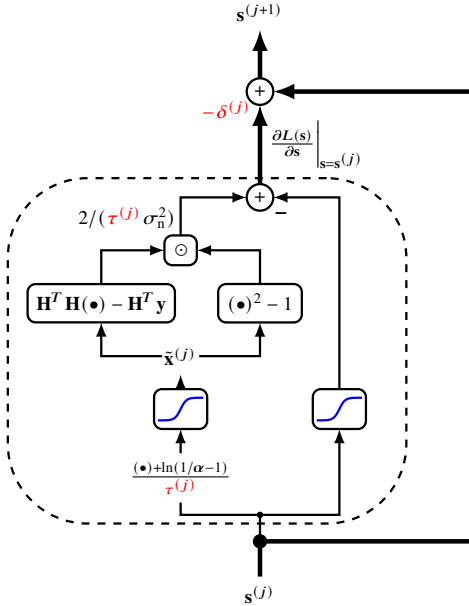


Fig. 3. One layer of the unfolded binary CMD algorithm. In red: trainable parameters.

TABLE I
SIMULATION SCENARIOS

Scenario	Sys Dim	Mod	Corr.	Coding
Large MIMO	32 × 32	QPSK	no	no
MIMO	8 × 8	QPSK	no	no
Multi-class	32 × 32	QAM16	no	no
massive MIMO One-Ring	32 × 64	QPSK	20°	no
Soft Output	32 × 32	QPSK	no	LDPC

with $\tilde{\mathbf{z}}_n = \sigma_{\tau(N_{it})}(\mathbf{g}_n^{(N_{it})})$ from the last iteration N_{it} of (14). To summarize, we optimize the parameter set θ of our approximating pdf $q(\mathbf{x}|\mathbf{y}, \theta)$ based on CMD:

$$\theta^* = \arg \min_{\theta} \mathbb{E}_{\mathbf{y} \sim p(\mathbf{y})} [\mathcal{H}(p(\mathbf{x}|\mathbf{y}), q(\mathbf{x}|\mathbf{y}, \theta))] \quad (32)$$

$$\approx \arg \min_{\theta} -\frac{1}{N} \sum_{i=1}^N \sum_{n=1}^{N_T} \left[\begin{array}{c} x_n = m_1 \\ \vdots \\ x_n = m_M \end{array} \right]^T \ln \left(\sigma_{\tau(N_{it})}(\mathbf{g}_n^{(N_{it})}) \right). \quad (33)$$

As a side effect, we also learn to relax with CMD by $\tau^{(j)}$. The optimization problem (33) can be efficiently solved by variants of Stochastic Gradient Descent (SGD). Thanks to having a model, we are able to create infinite training and test data for reasonable approximation with (33) in every iteration of SGD. We notice that this is in contrast to classic data sets from the machine learning community.

IV. NUMERICAL RESULTS

A. Implementation Details / Settings

In order to evaluate the performance of the proposed approach CMD, we present numerical simulation results for application in different MIMO systems with N_T transmit and N_R receive antennas given in Tab. I. We assume Rayleigh

TABLE II
ALGORITHMS

Abbrev.	Complexity	Literature
MAP / SD	$O(M^{\gamma N_T})$, $\gamma \in (0, 1]$	[20]
SDR	$O(\max(N_R, N_T)^3 N_T^{1/2} \log(1/\epsilon))$	[22]
OAMPNet	$O(N_L N_T^3)$	[33]
MMSE / MOSIC	$O(N_T^3)$	[34], [21]
DetNet	$O(N_L(N_T N_R + N_T^2 M))$	[27], [28]
MMNet (iid)	$O(N_L N_T(N_T + N_R + M))$	[34]
AMP	$O(N_{it} N_T(N_R + M))$	[24]
CMD	$O(N_L N_T(N_R + M))$	[37]
MF	$O(N_T N_R)$	

block fading and an uplink scenario where several UEs transmit to one BS. As an example, we assume the number of iterations or layers to be $N_{it} = N_L = N_T$. For numerical optimization of the parameters $\delta^{(j)}$ and $\tau^{(j)}$ of the unfolded CMD structure in (33), we employ the Tensorflow framework in Python [7]. Here, we use Adam (Adaptive Moment Estimation) as a popular variant of SGD with a default batch size of $N_b = 500$ and $N_{epoch} = 10^5$ training iterations. Although providing fast convergence and requiring little hyperparameter tuning, it is known to generalize poorly [38]. Since we are able to generate a sufficient amount of training data, i.e., $N = N_b \cdot N_{epoch} = 5 \cdot 10^7$ to fulfill (33) approximately, we make sure that generalization to unseen data points is possible. To allow for training in Tensorflow and for comparison to DNN-based approaches, we restrict to QAM constellations with Gray encoding and transform the complex-valued system model (1) into its real-valued equivalent so that we have $\mathbf{x} \in \mathcal{M}^{2N_T \times 1}$. The default range of training noise variance σ_n^2 is distributed uniformly between $E_b/N_0 = 10 \log_{10}(1/\sigma_n^2) - 10 \log_{10}(\log_2(M)) \in [4, 27]$. The default parameter starting point is set to θ_0 with constant $\delta_0^{(j)} = 1$ and heuristically motivated and linear decreasing

$$\tau_0^{(j)} = \tau_{\max} - (\tau_{\max} - 0.1)/N_{it} \cdot j \quad (34)$$

with $\tau_{\max} = 1/(M - 1)$, $j \in [0, N_{it}]$. With this choice, $p(\tilde{x})$ is always log-convex and hence reasonably approximating $p(x)$ (see Fig. 1). For training of DNN-based approaches DetNet and MMNet, we used the original implementations uploaded to GitHub (see [28], [34]) with only minor modifications to parametrization if beneficial. Consequently, we trained MMNet with CMD training SNR and layer number. Since we focus on offline derived or trained algorithms which are used for inference at run time, we used its i.i.d. variant. We always used the soft output version of DetNet with output normalization to 1 since we noted that performance is close to or better than the hard decision version. Furthermore, we compare CMD to several SotA approaches for MIMO detection (see Tab. II) choosing the number of Monte Carlo runs so that always 1000 errors are detected (100 for SD and SDR).

B. Symmetric channel

First, we test application of unfolded CMD in a large symmetric $32 \times 32 / 64 \times 64$ MIMO system with i.i.d. Gaussian

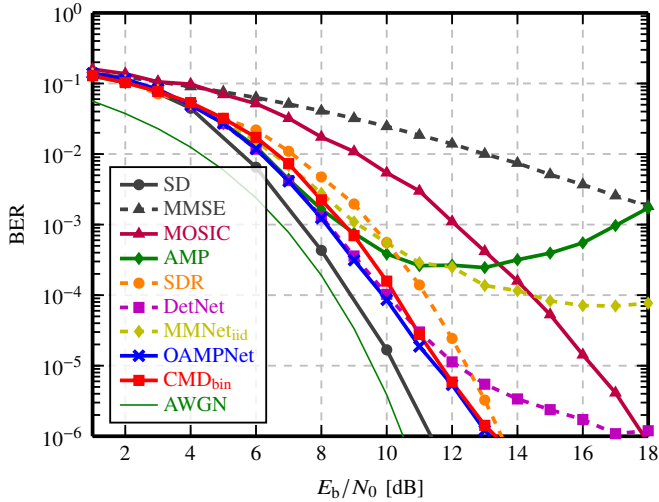


Fig. 4. BER curves of several detection methods in a 32×32 MIMO system with QPSK modulation. Effective system dimension is 64×64 and for iterative algorithms $N_{it} = N_L = 64$.

channel \mathbf{H} and QPSK/BPSK modulation. Fig. 4 shows the results in terms of Bit Error Rate (BER) as a function of E_b/N_0 . Owing to near-optimal performance, the SD is always provided as a benchmark in the following.

Linear detectors perform bad in this setup: Since the curve of the MF remains almost constant at $\text{BER} \approx 20\%$ and the Zero Forcer performs even worse, both are not shown in the following. At least, the MMSE equalizer shows acceptable behavior but is still separated by a 7 dB gap at $E_b/N_0 = 13$ dB from SD. In contrast, nonlinear SotA detectors like MOSIC from [21], AMP and SDR technique show good performance. Whereas AMP runs into an error floor for high SNR since the message statistics are not Gaussian anymore in finite small-dimensional MIMO systems, SDR proves to be a close relaxation by only dropping the non-convex requirement of $\text{rank}(\mathbf{x}\mathbf{x}^T) = 1$.

Notably, our approach CMD in its binary version CMD_{bin} performs very well comparable to DetNet and OAMPNet. Further, CMD_{bin} does not run into an error floor in the simulated SNR range like AMP and DetNet. Finally, we note that our approach is similar in complexity to AMP whereas DetNet and OAMPNet are very complex DNN architectures (see Tab. II). The other DNN-based approach MMNet has comparable low complexity but fails to beat CMD_{bin} . Since we observed an early error floor similar to AMP in all settings, we omit further results. We conjecture that the denoising layers are insufficient expressive in the interference limited high SNR region.

Results in a smaller 8×8 MIMO system plotted in Fig. 5, show that all soft non-linear approaches except for SDR and MOSIC run into an error floor at lower SNR. Thus, we conjecture that they share the same suboptimality at finite system dimensions. They may rely on the statistics of the interference terms to be Gaussian which is only approximately true for large system dimensions. Apart from SDR and MOSIC, CMD_{bin} offers still the best overall performance and is close in performance to SDR for $E_b/N_0 < 10$ dB.

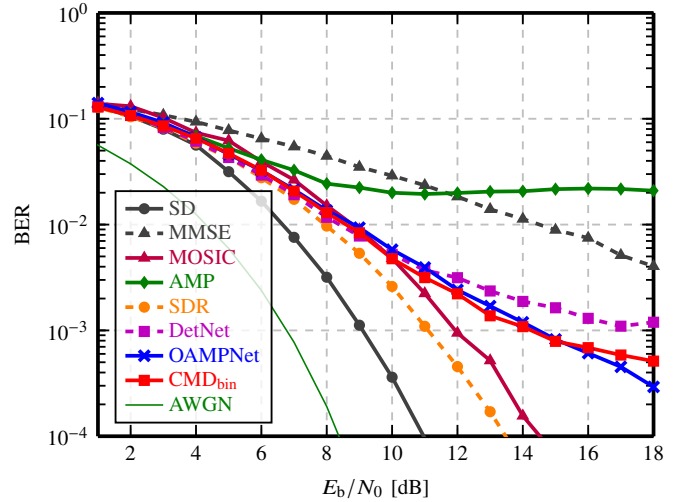


Fig. 5. BER curves of several detection methods in a 8×8 MIMO system with QPSK modulation. Effective system dimension is 16×16 and for iterative algorithms $N_{it} = N_L = 16$.

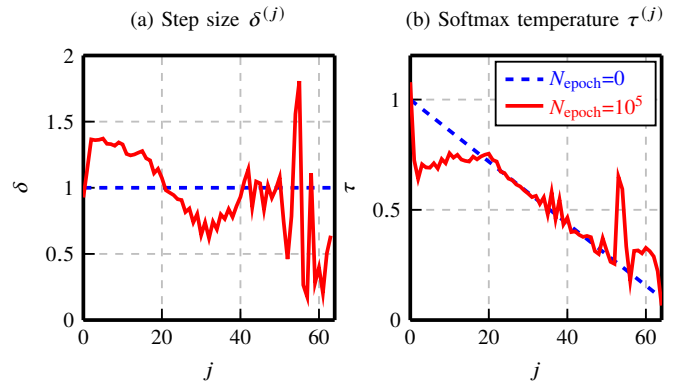


Fig. 6. Parameters θ of CMD_{bin} in a 32×32 MIMO system with QPSK modulation. Effective system dimension is 64×64 .

C. Algorithm and Parametrization

To investigate the influence of learning on unfolded CMD_{bin} and the values of its parameters θ , we visualize them per layer j in Fig. 6 for the 32×32 MIMO system considered before. We cannot observe any pattern after parameter optimization and interpretation seems very difficult.

Furthermore, we notice from Fig. 7 that starting point initialization θ_0 has a large impact on the optimum θ_{10^5} found by SGD (after $N_{\text{epoch}} = 10^5$ iterations). If we use a starting point $\theta_{0,\text{splin}}$ with linear decreasing

$$\theta_{0,\text{splin}}^{(j)} = \delta_{0,\text{splin}}^{(j)} = 1 - (1 - 0.01)/N_{it} \cdot j \quad (35)$$

for $j \in [0, N_{it}]$, a solution $\theta_{10^5,\text{splin}}$ is learned allowing CMD to perform better in the low E_b/N_0 region from 6 to 10 dB. CMD even reaches the performance of the best suboptimal algorithm considered in this setup OAMPNet but then runs into an error floor in contrast to default CMD. To explain this behavior in the interference limited higher E_b/N_0 region, we conjecture that a higher starting and correlating end step size (see Fig. 6) allows CMD to leave a local optimum with higher probability

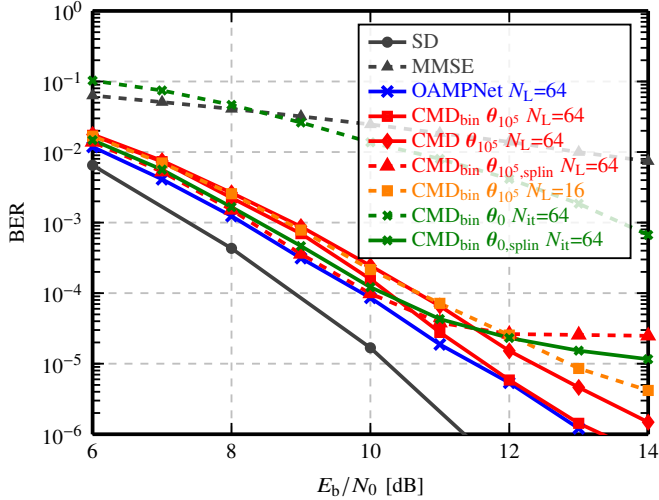


Fig. 7. BER curves of CMD with different parametrization or algorithmic in a 32×32 MIMO system with QPSK modulation. Effective system dimension is 64×64 .

and to find a better one whereas a small step size enforces convergence to a local solution. In the noise limited E_b/N_0 region, noise removal is crucial and hence convergence. This means CMD can be optimized for different working points and is sensitive to starting point initialization. Since CMD only has a small parameter set, we are able to load them dynamically and achieve very good performance in all E_b/N_0 regions.

In particular, we are able to further decrease the number of parameters with negligible performance loss. Unfolded CMD_{bin} with only $N_L = 16$ layers performs equally well compared to default CMD with $N_L = 64$ at low E_b/N_0 and slightly worse at $E_b/N_0 = 12$ dB by 1 dB.

Without unfolding, heuristics for parameter selection are required similar to starting point initialization. The detection performance with such heuristic parameters $\theta_{0,\text{splin}}$ is quite impressive since the BER curve is close to that of learned CMD with $\theta_{10^5,\text{splin}}$. Therefore, we are able to use the plain algorithm for detection. We note that this is not true with default parameters θ_0 and that performance can be quite different after optimization (θ_{10^5}).

Finally, we compare the performance of algorithm CMD_{bin} for the special case of binary RV with that of the generic multi-class algorithm CMD since both are different. From Fig. 7, we observe that the performance is very similar and conjecture that CMD is capable of achieving the same performance if training is parameterized correctly.

D. Multi-class Detection

So far, only BPSK modulation and hence two classes have been considered. To test multi-class detection with $M = 4$ classes, we show numerical results in a 32×32 MIMO system with 16-QAM modulation being equivalent to a 64×64 4-ASK MIMO system after transformation into the equivalent real-valued problem. Owing to now 3 degrees of freedom in the soft-max function and denser symbol packing, we changed our batch size to $N_b = 1500$ and training SNR to higher $E_b/N_0 \in$

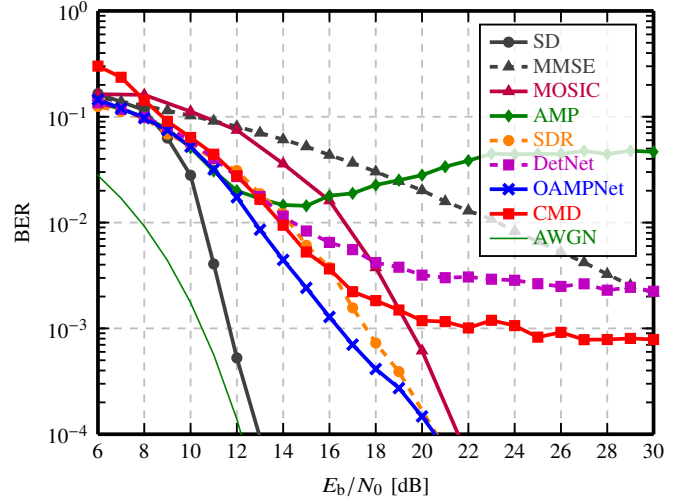


Fig. 8. BER curves of several detection methods in a 32×32 MIMO system with 16-QAM modulation. Effective system has dimension 64×64 and 4-ASK modulation and for iterative algorithms $N_{\text{it}} = N_L = 64$.

[10, 33], respectively. Setting the default starting point with $\tau_{\text{max}} = 2/(M - 1) = 2/3$ so that the MAP criterion $\ln p(\tilde{\mathbf{x}}, \mathbf{y})$ becomes convex for a couple of iterations proves to be crucial for successful training of CMD with multiple classes.

Fig. 8 shows BER curves in this system. Clearly, we can now observe a large gap between the BER curve of SD and that of all other suboptimal approaches. Comparing suboptimal algorithms, OAMPNet is superior over the whole SNR region. Observing a maximum 2 dB curve shift, we note that CMD is competitive to OAMPNet and SDR at $E_b/N_0 \in [10, 17]$ and when $\text{BER} = [10^{-2}, 10^{-3}]$ which is a typical working point of decoders. Without training parameter tuning, CMD performs even worse than the MMSE equalizer. At higher SNR, an error floor follows. Although using a more expressive DNN model, DetNet now trained for $E_b/N_0 \in [9, 16]$ fails to beat CMD especially in this region.

E. Massive MIMO

Investigation in large symmetric MIMO systems reveals the potential and shortcomings of the algorithms. Rather in 5G, massive MIMO systems with $N_R > N_T$ are employed [19]. Assuming i.i.d. Gaussian channels, we shortly report the results of a 32×64 MIMO system with QPSK modulation: The BER curves of learning based approaches and SDR almost follow that of SD and thus suggest that they fit perfectly for application in massive MIMO.

However in practice, channels are spatially correlated at the receiver side due to good spatial resolution of BS' large arrays compared to the number of scattering clusters [19]. Hence, the results for i.i.d. Gaussian channel are less meaningful as noted in [34]. As a first and quick attempt towards a realistic channel model which captures its key characteristics, we test performance in the so-called One-ring model assuming a BS equipped with a uniform linear antenna array [28], [19]. We parameterize the correlation matrices of every column in \mathbf{H} with reasonable values: Assuming an urban cellular network,

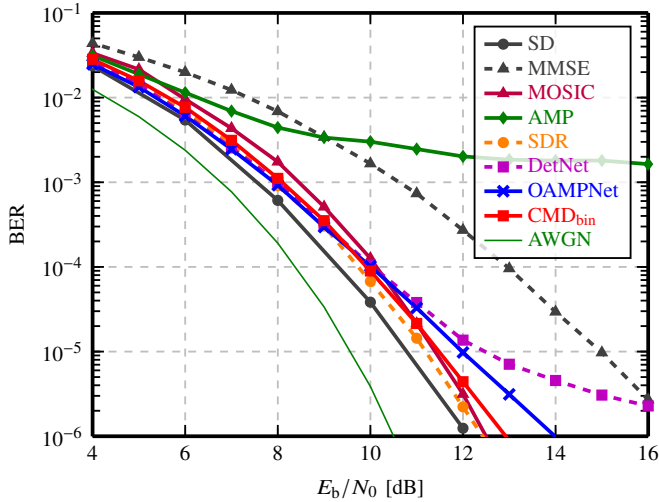


Fig. 9. BER curves of several equalization methods in a correlated 32×64 MIMO system with QPSK modulation. The correlation matrices were generated according to a One-Ring model with 20° angular spread and 120° cell sector. Effective system dimension is 64×128 and for iterative algorithms $N_{it} = N_L = 64$.

we set the angular spread to 20° and sample the nominal angle uniformly from $[-60^\circ, 60^\circ]$, i.e., 120° cell sector.

From Fig. 9, it becomes evident that the performance loss of learning based approaches compared to SD in such a One-Ring model of dimension 32×64 is similar to the symmetric setting 32×32 . Surprisingly, MOSIC and SDR now prove to be comparable whereas the BER of AMP degrades. A sequential schedule with SNR ordering seems to be beneficial in the One-Ring model. CMD performs very close to DetNet and OAMPNet below 10 dB and even outperforms the latter above. Hence, it proves to be a generic and hence promising detection approach.

F. Soft Output (Coded System)

After investigation of detection performance in uncoded systems, we turn to an interleaved and horizontally coded $32 \times 32 / 64 \times 64$ MIMO system with Rayleigh block fading reflecting our uplink model. We aim to verify whether not only hard decisions but also soft outputs generated by unfolded CMD and the soft output version of DetNet have high quality. This is especially important in practice since coding is an essential component besides equalization in today's communication systems. Therefore, we use a 64×128 LDPC code with rate $R_C = 1/2$ from [39] and at receiver side a belief propagation decoder with 10 iterations. The results in terms of Coded Frame Error Rate (CFER) as a function of $E_b/N_0/R_C$ are shown in Fig. 10. Owing to overwhelming computational complexity, we refrained from using the MAP solution with coding as a benchmark and instead show uncoded CMD and SD curves for reference. Strikingly, CMD with coding beats the latter and allows for a coding gain. In contrast, AMP with coding runs into an error floor after 9 dB: The output statistics become unreliable for high SNR in finite dimensional systems. Surprisingly, although being one of the best detection methods in the uncoded setting, DetNet with coding performs close to

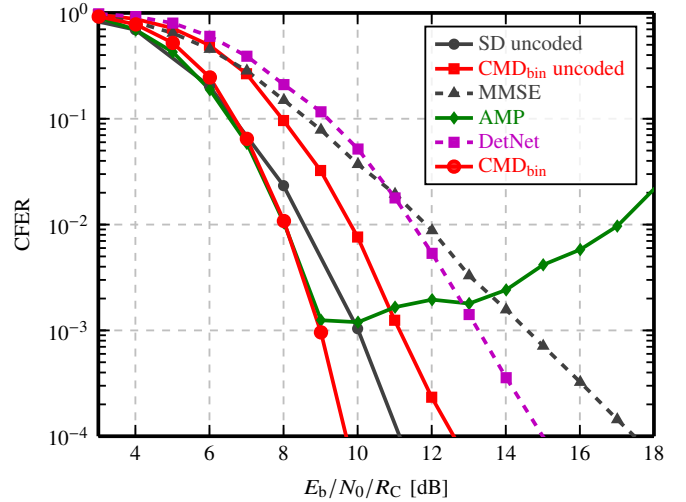


Fig. 10. CFER curves of a horizontally coded 32×32 MIMO system with QPSK modulation. A 64×128 LDPC code with belief propagation decoder was used. Effective system dimension is 64×64 and for iterative algorithms $N_{it} = N_L = 64$.

MMSE equalization with soft outputs and thus worse than expected. Actually, the soft output version of DetNet should deliver accurate probabilities or Log Likelihood Ratios (LLRs) according to [28] after optimization.

Indeed, we visualize with an exemplary histogram of LLRs that this is not the case. In Fig. 11, we show the relative frequencies of LLRs of one symbol x_n in one random channel realization \mathbf{H} for $E_b/N_0 = 10$ dB. First, we note the histograms for $x_n = -1$ and $x_n = 1$ to be symmetric meaning that both algorithms fulfill a basic quality criterion. Furthermore, it can be clearly seen that DetNet mostly provides hard decisions with $\approx 97\%$ LLRs being $-\infty$ and ∞ , respectively. Only a few values are close to 0. In contrast, CMD provides meaningful soft information resembling a mixture of Gaussians as expected from literature [40] ranging from -30 to 30 . These results strongly indicate that the difference of soft output quality originates from different underlying optimization strategies: As pointed out in Section III-B, unfolded CMD relies on minimization of KL divergence between IO a-posteriori and approximating pdf whereas the one-hot representation in DetNet is optimized w.r.t. MSE. We conclude that our approach yields a better optimization strategy.

G. Complexity Analysis

Since complexity is the main driver for development of suboptimal algorithms like CMD instead of relying on MAP detection, we complete our numerical study by relating detection performance to results on the computational complexity given in Tab. II. With regard to CMD, the iterative asymptotic complexity of $O(N_T(2N_R+4M))$ or $O(2N_TN_R)$ for binary RV is dominated by the matrix vector multiplications in $\mathbf{H}^T \mathbf{H} \tilde{\mathbf{x}}$, i.e., CMD scales linearly with the input and output dimension as well as the number of classes. Clearly, CMD has very low complexity comparable to AMP and MMNet but with remarkable higher detection rate.

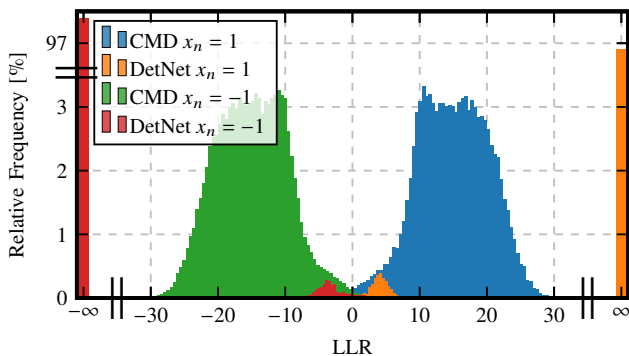


Fig. 11. Exemplary histogram showing the relative frequencies of LLRs of one symbol x_n in one random channel realization \mathbf{H} at $E_b/N_0 = 10$ dB.

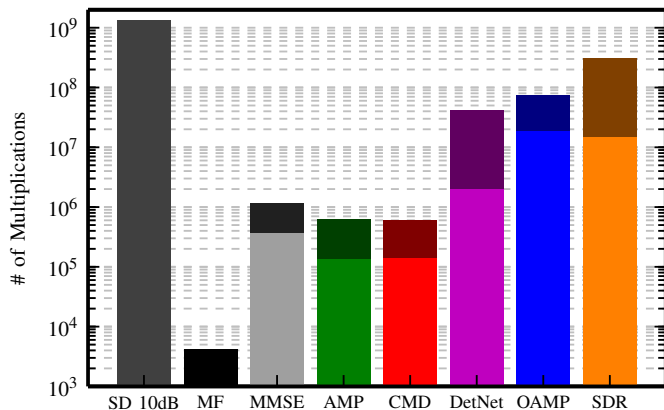


Fig. 12. Complexity of detection algorithms in terms of number of multiplicative operations in a $32 \times 32 / 64 \times 64$ MIMO system: Light colored bars indicate a realistic low-complexity implementation with BPSK and dark colored bars the worst-case complexity with 16QAM modulation.

Besides qualitative $O(\cdot)$ analysis, we capture complexity quantitatively by counting the number of Multiplicative Operations (MOPs) for one iteration and channel realization being the most common and costly floating point operations. Fig. 12 shows the respective bar chart assuming a realistic low-complexity implementation in a 32×32 with QPSK ($M = 2$) and $N_L = 16$ and worst-case complexity implementation with 16QAM modulation ($M = 4$) and $N_L = 64$, respectively. For BPSK and the lower bar of MMSE equalization, we assumed Gaussian elimination to solve the linear equation system and, for higher order QAM and the higher bar, LU decomposition. We estimate the upper bound on SDR MOP count by unadapted $O(\max(N_R, N_T)^4 N_T^{1/2} \log(1/\epsilon))$ and the lower bound on MOPs to account for half of the FLOPS from [28] with inaccurate $\epsilon = 0.1$. The expected number of visiting nodes $O(M^\gamma N_T)$ of the SD is SNR dependent with $\gamma \in (0, 1]$ and was extracted from [20].

Apparently, only the very basic MF beats CMD at considerably worse detection performance. Finally, we note that AMP, MMNet, DetNet and CMD further come with the benefit of already delivering soft outputs. But in contrast, MMSE and MOSIC are able to reuse its computations with only one matrix vector multiplication remaining for any further detection inside the channel coherence time interval.

V. CONCLUSION

In this article, we introduced the so called continuous relaxation of discrete RV to the MAP detection problem. Allowing to replace exhaustive search by continuous optimization, we defined our classification approach Concrete MAP Detection (CMD), e.g., based on gradient descent. By unfolding CMD into a DNN, we further were able to optimize its low number of parameters and hence to improve detection performance while limiting it to low complexity. As a side effect, the resulting structure allows for fast online training. Using the example of MIMO detection, simulations reveal CMD to be a generic detection method competitive to SotA outperforming other recently proposed ML-based approaches DetNet and MMNet in every considered scenario and in terms of complexity. Notably, we selected an optimization criterion grounded in information theory, i.e., cross entropy, and showed that it aims at learning an approximation of the individual optimal detector. By simulations in coded systems, we demonstrated its ability to provide reliable soft outputs as opposed to [28], being a requirement for soft decoding, a crucial component in today's communication systems.

All these findings prove CMD to be a promising detection approach for application in future massive MIMO systems. Further research is required to improve the performance with higher-order modulations and to demonstrate its applicability to non-linear scenarios of other research domains.

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