# Optimum Quantization of Memoryless Channels with $N$-ary Input

Tobias Monsees, Dirk Wübben and Armin Dekorsy
Department of Communications Engineering
University of Bremen, 28359 Bremen, Germany
Email: {tmonsees, wuebben, dekorsy}@ant.uni-bremen.de

***Abstract*—This paper considers channel quantization of memoryless channels with $N$-ary input $x$ and Mutual Information (MI) as fidelity criterion. We make use of an equivalent formulation of the quantization problem that transforms the channel output $y$ into a $N-1$ dimensional probability-simplex by using the posterior-distribution $p(x|y)$. By using Burshtein's optimality theorem, it is possible to show that there exist an optimal solution that is obtained by separating hyperplane cuts in this probability-simplex. We show that for practically relevant real valued input/output channels, the posterior-distribution $p(x|y)$ is located on a *smooth* curve in the $N-1$ dimensional probability-simplex. Under mild conditions, the optimality theorem provides the existence of an optimal solution that is obtained by separating connected segments of this curve. For this case, we provide further insights into the underlying optimization problem and motivate a Dynamic Programming (DP) approach for finding the global optimal quantizer mapping that maximizes the end-2-end MI for the given cardinality of the quantizer output. Numerical investigation with $N$-ASK input and real valued Additive White Gaussian Noise (AWGN) show that this approach is superior to common design approaches which only converge to a local optimal quantizer mapping.**

***Index Terms*—Quantization, mutual information, dynamic programming, memoryless channels**

## I. Introduction

Quantization plays a significant role in digital communications, signal processing, lossy data compression and information theory [1]. In lossy data compression, quantization is commonly used to minimize a predefined distortion measure between the input and the output of the quantizer (e.g. mean square error (MSE) [2]). In this case, the fundamental Rate-Distortion (RD) theory [3] quantifies the trade-off between compression and distortion. In digital communications, the received signals are corrupted (noisy) versions of the original source signal(s) and transformed via analog-to-digital converters (ADCs) into discrete amplitude signals for further processing. Reducing the bit-resolution of the resulting amplitude quantizer limits the power consumption and hardware costs of the communication system [4], but sacrifices the communication performance. In terms of the ADC bit-resolution, common figure of merits assume that each added bit at least doubles the power dissipation [5] of the ADC.

To minimize the impact on the communication performance, quantization schemes with low bit-resolution require optimization of the quantization thresholds [6] to maximize the e2e Mutual Information (MI) $I(\mathsf{x};\mathsf{z})$ between the source signal $\mathsf{x}$ and quantizer output $\mathsf{z}$. Jointly maximizing MI in both, the input distribution and the quantizer mapping is a concave-convex optimization problem that is known to be NP-hard in general, i.e., polynomial time algorithms to find the solution only exist for special cases [7]. However, if the source distribution is fixed, finding the quantizer that maximizes the e2e MI $I(\mathsf{x};\mathsf{z})$ will give the highest achievable rate. The relation between MI maximizing quantization and the Information Bottleneck Method (IBM) has been pointed out in [8]. A detailed investigation on the connection of the IBM to coding and learning theory is provided in [9]. A comprehensive study of heuristic algorithms to tackle the underlying optimization problem is provided in [10]. In the recent literature, MI maximizing quantization has been successfully applied for the design of discrete receivers [11], [12] and discrete decoders with very low bit resolution for Low Density Parity Check (LDPC) codes [13]–[15] as well as polar codes [16].

For the special case of binary-input discrete memoryless channels (DMCs), Yagi and Kurkoski proposed an algorithm based on Dynamic Programming (DP) with the random coding exponent as design criteria, which includes the cut-off rate and the MI as special cases [17]. In [18], the optimality proof of this approach for the design of quantizers for the general DMC with binary-input maximizing the MI was given. The algorithmic approach has been generalized to maximize $\alpha$-MI with an efficient implementation based on the SMAWK algorithm [19], [20]. Furthermore, He et al. extended this approach to non-binary input DMCs and even more general concave cost functions [21]. As already mentioned in [21], finding a general condition on the DMC to ensure that the DP approach will find the global optimal solution is an open problem.

In this paper[1], we investigate the problem of quantizing DMCs with $N$-ary input. We show that for real valued $N$-ary sources and *friendly* communication channels, the Sequential Deterministic Quantizer (SDQ) design via DP delivers a global optimal solution with low run time complexity. In contrast to previous works, we deduce the optimality of the DP approach by the relation to an equivalent centroid based clustering problem.

[1]Notation: Random variables are denoted by sans-serif letters $\mathsf{x}$, random vectors by bold sans-serif letters $\mathbf{\mathsf{x}}$, realizations by serif letters $x$ and vector valued realizations by bold serif letters $\boldsymbol{x}$. Sets are indicated by calligraphic letters $\mathcal{X}$ and the distribution of a random variable $\mathsf{x}$ is given by $p(x)$.

The remainder of this paper is structured as follows: Section II introduces the system model. The quantizer design problem is discussed in Section III and its relation to statistical learning in Section IV. Section V describes the SDQ design approach for $N$-ask input and real valued communication channels and provides numerical evaluations of the performance. Finally, Section VI concludes this work.

## II. System Model

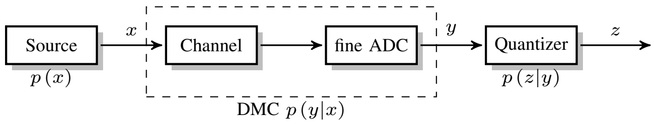


Fig. 1: System model for quantizer design

The system model for the quantizer design is shown in Fig. 1. The source $\mathsf{x}$ takes realizations $x$ on a discrete set $x \in \mathcal{X} = \{x_1, x_2, ..., x_N\}$ of $N$ symbols with probability distribution $p(x)$. Transmitting the source $\mathsf{x}$ over a physical channel results in a continuous channel output. Without loss of generality, a fine ADC converts the channel output into a discrete random variable $\mathsf{y}$ that takes realizations $y$ on the discrete set $\mathcal{Y} = \{y_1, y_2, ..., y_{N_\mathsf{y}}\}$ of finite cardinality $N_\mathsf{y}$. The transition probability distribution of the resulting DMC is denoted as $p(y|x)$. The DMC output $\mathsf{y}$ is quantized into $N_\mathsf{z}$ clusters $z \in \mathcal{Z} = \{z_1, z_2, ..., z_{N_\mathsf{z}}\}$ via a quantizer mapping $Q(y)$ that is uniquely determined by the transition probability distribution $p(z|y)$. Thus, the quantizer optimization yields also quantization regions for the continuous output channel. The successive blocks in Fig. 1 induce a Markov chain $\mathsf{x} \rightarrow \mathsf{y} \rightarrow \mathsf{z}$, i.e., $\mathsf{x}$ and $\mathsf{z}$ are conditionally independent given $\mathsf{y}$.

## III. Information Maximizing Quantizer Design

The objective of the quantizer is to maximize the e2e MI $I(\mathsf{x};\mathsf{z})$. The resulting optimization problem is to find the distribution $p^\star(\mathsf{z}|\mathsf{y})$ which maximizes the e2e MI $I(\mathsf{x};\mathsf{z})$ between the source $\mathsf{x}$ and the quantizer output $\mathsf{z}$, i.e.

$$p^\star(\mathsf{z}|\mathsf{y}) = \underset{p(\mathsf{z}|\mathsf{y}) \in \mathcal{Q}}{\operatorname{argmax}} \; I(\mathsf{x};\mathsf{z}) \;\; \text{s.t.} \;\; |\mathcal{Z}| = N_\mathsf{z} < N_\mathsf{y} \,. \tag{1}$$

In contrast to MSE based quantizer design, it is important to notice that (1) is independent of the numerical values in the set $\mathcal{Z}$ since MI depends only on distribution functions, i.e., we can define the values in $\mathcal{Z}$ by any convenient choice (e.g. integers) without limitations on the e2e MI $I(\mathsf{x};\mathsf{z})$. Since, the probability of an individual pair $z, y \in \mathcal{Z} \times \mathcal{Y}$ is $p(\mathsf{z} = z|\mathsf{y} = y) \in [0, 1]$, the conditional probability distribution $p(\mathsf{z}|\mathsf{y} = y)$ for an individual realization $y \in \mathcal{Y}$ is a point in the $N_\mathsf{z} - 1$ dimensional probability simplex[2], since $\sum_{z \in \mathcal{Z}} p(\mathsf{z} = z|\mathsf{y} = y) = 1$. The resulting search set $\mathcal{Q}$ in (1) for the distribution $p(\mathsf{z}|\mathsf{y})$ is the $N_\mathsf{y}$ fold Cartesian product of the $N_\mathsf{z} - 1$ dimensional probability simplex.

[2]The $N - 1$ dimensional probability simplex is defined by $\Delta_{N-1} = \{\boldsymbol{\theta} \in \mathbb{R}^N \mid \sum_{i=1}^N \theta_i = 1 \text{ and } 0 \leq \theta_i \leq 1 \text{ for } i = 1, ..., N\}$.

The maximization problem in (1) belongs to class of *concave* programming problems. This implies the existence of a global optimal solution at an extreme point of the set $\mathcal{Q}$ [18], i.e., a globally optimal *deterministic* quantizer mapping $p^\star(z|y) \in \{0, 1\}$ for all $z, y \in \mathcal{Z} \times \mathcal{Y}$ exist. However, finding the global optimal solution via a naive brute force search over all possible Deterministic Quantizers (DQs) $\mathcal{Q}_{\text{DQ}}$ has exponential complexity in the number of output values, since $|\mathcal{Q}_{\text{DQ}}| = N_\mathsf{z}^{N_\mathsf{y}}$ possible DQs exist. Common algorithmic approaches converge only to a local optimal solution [10] and the quality of the resulting solution is often difficult to interpret since the performance loss compared to the global optimal solution might be very large. For a DQ, the set of output values $y$ that are mapped into a cluster $z \in \mathcal{Z}$ is denoted as pre-image

$$\mathcal{Y}_z = \{y \in \mathcal{Y} \mid Q(y) = z\} \,. \tag{2}$$

## IV. Quantizer Design via Centroid based Clustering

### A. Reformulation of the Quantizer Design Problem

The optimization problem in (1) can be interpreted as a clustering problem [22] with Kullback-Leibler (KL) divergence[3] as similarity measure. The equivalent interpretation exploits the Markov chain condition and relies on the property that the e2e MI $I(\mathsf{x};\mathsf{z})$ is expressed as [22]

$$I(\mathsf{x};\mathsf{z}) = I(\mathsf{x};\mathsf{y}) - \underbrace{\mathrm{E}_{\mathsf{y},\mathsf{z}}\left\{D_{\mathrm{KL}}(p(\mathsf{x}|\mathsf{y}) \,\|\, p(\mathsf{x}|\mathsf{z}))\right\}}_{L(Q)} \,. \tag{3}$$

Since $I(\mathsf{x};\mathsf{y})$ is fixed, maximizing the e2e MI $I(\mathsf{x};\mathsf{z})$ in (1) is equivalent to finding the DQ $Q^\star$ that minimizes the expected KL loss

$$L(Q) = \mathrm{E}_{\mathsf{y},\mathsf{z}}\left\{D_{\mathrm{KL}}(p(\mathsf{x}|\mathsf{y}) \,\|\, p(\mathsf{x}|\mathsf{z}))\right\} \,, \tag{4}$$

between the two A Posteriori Probability (APP) distributions $p(\mathsf{x}|\mathsf{y})$ and $p(\mathsf{x}|\mathsf{z})$. The data processing inequality [3] implies that the KL loss $L(Q)$ is always non-negative, i.e., $L(Q) \geq 0$. Hence, we can define an APP vector in the posterior probability space for each observation $y \in \mathcal{Y}$ as

$$\boldsymbol{f}_y = [p(x_1|y), ..., p(x_N|y)] \in \Delta_{N-1} \,, \tag{5}$$

and an APP vector for each quantizer output $z \in \mathcal{Z}$ by

$$\boldsymbol{g}_z = [p(x_1|z), ..., p(x_N|z)] \in \Delta_{N-1} \,. \tag{6}$$

Notice that the transformations in (5) and (6) results in two vector valued random variables $\boldsymbol{f}_\mathsf{y}$ and $\boldsymbol{g}_\mathsf{z}$, both taking realizations in the $N - 1$ dimensional probability simplex $\Delta_{N-1}$. Using the representation $\boldsymbol{f}_y$ for the observations $y$, we can define an equivalent quantizer $\tilde{Q}$ that maps each APP vector $\boldsymbol{f}_y$ to one APP vector $\boldsymbol{g}_z$. The pre-images for $z \in \mathcal{Z}$ of this equivalent quantizer are

$$\mathcal{F}_z = \{\boldsymbol{f}_y \mid \tilde{Q}(\boldsymbol{f}_y) = z\} \,. \tag{7}$$

[3]The relative entropy or KL divergence between two distributions $p$ and $q$ is defined as $D_{\mathrm{KL}}(p\|q) = \sum_{x \in \mathcal{X}} p(x) \log_2 \frac{p(x)}{q(x)}$

The quantizer $\tilde{Q}$ has always a corresponding quantizer $Q$ in a sense that $Q(y) = \tilde{Q}(\boldsymbol{f}_y) = z$ for all $y, z \in \mathcal{Y} \times \mathcal{Z}$. The following theorem is based on the result for minimum impurity partitions from statistical learning theory [23].

*Theorem 1 (Separating Hyperplane):* [18] There exist a Deterministic Quantizer (DQ) $\tilde{Q}^\star : \Delta_{N-1} \to \mathcal{Z}$, where every pair of different pre-images $\mathcal{F}_z$ and $\mathcal{F}_{z'}$ are separated by a hyperplane in the posterior probability simplex $\Delta_{N-1}$ and the corresponding Deterministic Quantizer (DQ) $Q^\star$ maximizes $I(\mathsf{x};\mathsf{z})$ in (1).

This theorem implies the existence of a DQ $\tilde{Q}^\star$ where the convex hulls of every pair of different pre-images have an empty intersection. However, finding the globally optimal solution in a general setting, i.e., without any restrictions on the input or the DMC, is an NP-hard problem [24].

### B. KL-Means Algorithm

A popular approach for centroid based clustering is the KL-means algorithm [25]. It starts with a random initialization of cluster centroids $\boldsymbol{g}_z$ and iterates between clustering points $\boldsymbol{f}_y$ into bins with the same *nearest* centroid (assignment step)

$$\mathcal{F}_z = \left\{\boldsymbol{f}_y \mid D_{\mathrm{KL}}\left(\boldsymbol{f}_y \| \boldsymbol{g}_z\right) \le D_{\mathrm{KL}}\left(\boldsymbol{f}_y \| \boldsymbol{g}_{z'}\right), z \neq z'\right\} \tag{8}$$

and recalculation of the mean per cluster (update step)

$$\boldsymbol{g}_z = \frac{\sum_{\boldsymbol{f}_y \in \mathcal{F}_z} \boldsymbol{f}_y \cdot p\left(\boldsymbol{f}_y\right)}{\sum_{\boldsymbol{f}_y \in \mathcal{F}_z} p\left(\boldsymbol{f}_y\right)} \tag{9}$$

until either a maximum number of iterations is reached or a convergence criterion is fulfilled. The KL-means algorithm is equivalent to the Information Bottleneck (IB) algorithm for $\beta \to \infty$ [22].

Fig. 2 visualizes the clustering result in case of a 3-PSK source $\mathsf{x}$ with uniform input distribution $p(x)$ transmitted over a complex Additive White Gaussian Noise (AWGN) channel. The black line (—) represents the boundary of the 2-dimensional probability simplex $\Delta_2$ and the three black dots are its extreme points. The color of a point $\boldsymbol{f}_y$ indicates that it belongs to a cluster $z = \tilde{Q}(\boldsymbol{f}_y)$ with centroid $\boldsymbol{g}_z$ (red dot).

### C. Binary Input with General DMC

If the input is binary, the probability simplex $\Delta_1$ is a line. Furthermore, we can assume that the elements in $\mathcal{Y}$ satisfy

$$p\left(x_2|y_1\right) < p\left(x_2|y_2\right) < \ldots < p\left(x_2|y_{N_\mathsf{y}}\right) . \tag{10}$$

This assumption is without loss of generality, since the output $y \in \mathcal{Y}$ can always be re-labeled such that (10) holds. Strict inequality can always be ensured since if two elements have the same posterior probability, they have the same coordinates in the simplex (as defined in (5)) and hence they can be merged into one single element without any influence on the e2e MI $I(\mathsf{x};\mathsf{z})$. The condition (10) implies that the likelihood-ratios satisfy

$$\frac{p\left(y_i|x_2\right)}{p\left(y_i|x_1\right)} < \frac{p\left(y_{i+1}|x_2\right)}{p\left(y_{i+1}|x_1\right)} \quad \forall i \in \{1, \ldots, N_\mathsf{y}-1\} . \tag{11}$$

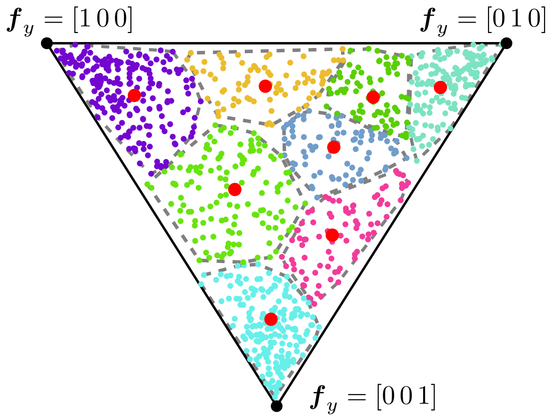


Fig. 2: Clustering result of the KL-Means algorithm with $N_\mathsf{z} = 8$ clusters in case of a 3-PSK source transmitted over a Quantized Additive White Gaussian Noise (QAWGN) channel with noise variance $\sigma_\mathsf{n}^2 = 1$

For $N = 2$, the APP vector of each realization $y_i$ is given $\boldsymbol{f}_{y_i} = [p(x_1|y_i)\ 1 - p(x_1|y_i)]$ such that all points $\boldsymbol{f}_y$ are located on a *straight line* between $\boldsymbol{f}_y = [1\,0]$ and $\boldsymbol{f}_y = [0\,1]$. Inequality (10) ensures that neighbouring points $\boldsymbol{f}_y$ on that line have a *similar meaning* w.r.t. the source signal $\mathsf{x}$.

Based on Theorem 1 the global optimal DQ clusters points $\boldsymbol{f}_y$ with similar meaning and for $N = 2$ the corresponding hyperplanes that separate two different pre-images are given by simple thresholds (points) on the *straight line*. With $y_{b_{z-1}}$ and $y_{b_z}$ denoting theses quantizer boundaries for a specific cluster $z \in \mathcal{Z}$, the corresponding pre-image $\mathcal{F}_z$ contains the *sequentially* indexed elements $\boldsymbol{f}_y$, i.e.,

$$\mathcal{F}_z = \left\{\boldsymbol{f}_{y_{b_{z-1}+1}}, \boldsymbol{f}_{y_{b_{z-1}+2}}, \ldots, \boldsymbol{f}_{y_{b_z}}\right\} . \tag{12}$$

Obviously, it is sufficient to search for all boundaries $y_{b_z}$ in order to find the global optimal quantizier. Such quantizer with sequentially labeled elements per pre-image is denoted as Sequential Deterministic Quantizer (SDQ) [18], [21]. To find the global optimal quantizer, the authors of [18] proposed a state-based algorithm to search over all possible SDQs. The algorithm is guaranteed to find the global optimal solution and can be interpreted as an instance of Dynamic Programming (DP). The complexity of this algorithm is $\mathcal{O}(N_\mathsf{z}(N_\mathsf{y} - N_\mathsf{z})^2)$ which can be reduced to $\mathcal{O}(N_\mathsf{z}(N_\mathsf{y} - N_\mathsf{z}))$ by the application of the SMAWK algorithm [20].

Fig. 3 shows the posterior probability distribution $p(\mathsf{x}|\mathsf{y})$ in case of a BPSK source transmitted over an AWGN channel. The output is uniformly discretized into $N_\mathsf{y} = 128$ clusters. The resulting bins are labeled successively by the mean value of its boundaries. For this finely quantized AWGN channel, the resulting DMC output $y \in \mathcal{Y}$ follows already the ordering of posterior probabilities in (10). The DMC is quantized into $N_\mathsf{z} = 8$ clusters and the resulting quantization regions that are found via the DP approach are indicated by different colors. The coloured points $\boldsymbol{f}_y$ in the simplex $\Delta_1$ demonstrate that neighboring points with similar meaning are clustered together.

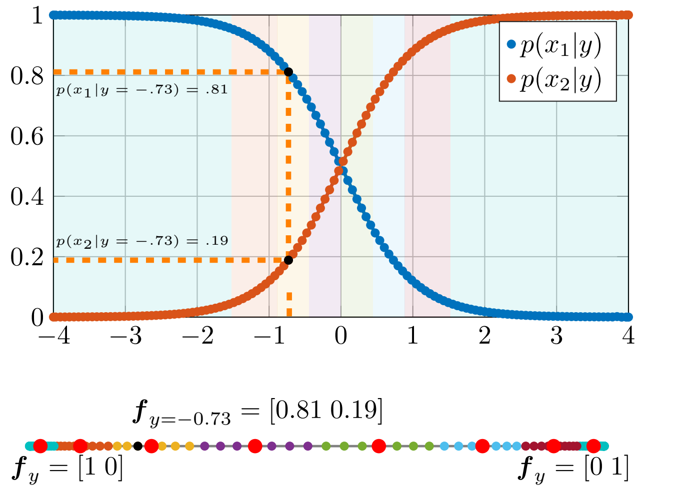


Fig. 3: (top) Posterior distribution $p(\mathsf{x}|\mathsf{y})$ for BPSK source transmitted over a QAWGN channel. (bottom) Visualization of posterior distribution $p(\mathsf{x}|\mathsf{y})$ in the probability simplex $\Delta_1$.

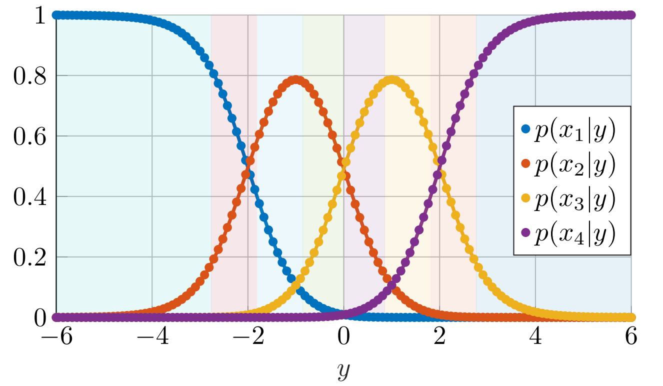


Fig. 4: Posterior probability distribution $p(\mathsf{x}|\mathsf{y})$ of a 4-ASK source (i.e. $x \in \mathcal{X} = \{-3, -1, 1, 3\}$) and transmission over a finely QAWGN channel

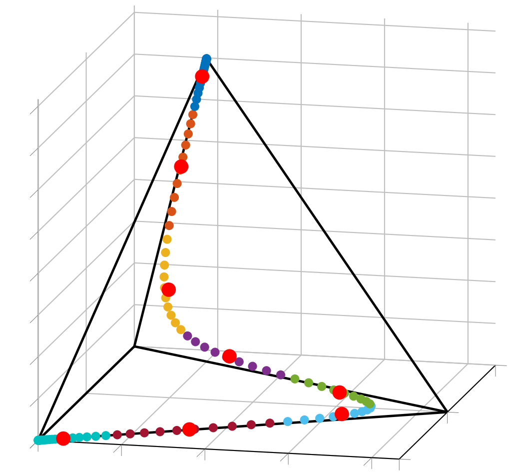

Fig. 5: Probability simplex $\Delta_3$ with points $\boldsymbol{f}_y$ and clusters with centroid coordinates $\boldsymbol{g}_z$

## V. Optimal Sequential Quantizer for $N$-ary Input

For $N > 2$, a one dimensional ordering of posterior probabilities as in (10) cannot be established for general DMCs. However, if the DMC is obtained by fine pre-quantization of a typical communication channel (e.g. AWGN) with real input and output, the observations $y$ have a *natural ordering* w.r.t. the transmit symbols $x \in \mathcal{X}$, i.e

$$\frac{p(y_i|x_j)}{p(y_i|x_1)} < \frac{p(y_{i+1}|x_j)}{p(y_{i+1}|x_1)} \quad \forall i \in \{1, ..., N_\mathsf{y} - 1\} \qquad (13)$$

and $\forall j \in \{2, ..., N\}$. In this case, we observe that points $\boldsymbol{f}_y$ with similar *meaning* are sequentially located on a *smooth curve* in the probability simplex $\Delta_{N-1}$. The optimal quantizer is again an SDQ and can be found via the same DP approach.

To visualize the resulting quantization problem, we consider a 4-ASK source, i.e., $\mathcal{X} = \{\pm 1, \pm 3\}$ with uniform source distribution $p(x)$. Again, the continuous output AWGN channel with noise variance $\sigma_\mathsf{n}^2 = 1$ is converted into a QAWGN channel by uniform pre-quantization into $N_\mathsf{y} = 128$ clusters. The posterior distribution $p(\mathsf{x}|\mathsf{y})$ is shown in Fig. 4. Moreover, Fig. 5 visualizes the 3-dimensional probability simplex $\Delta_3$ with mappings $\boldsymbol{f}_y$ and $\boldsymbol{g}_z$, showing that successive elements $y \in \mathcal{Y}$ are also successive elements on a *smooth curve*. In this example, the optimal SDQ is optimal among all possible DQs and the DP approach is utilized to find this solution. The resulting quantization regions are indicated by different colors.

It is important to notice that the optimal SDQ design is not optimal for complex input/output channels or artificial channels since a natural ordering as in (13) cannot be ensured.

### *Performance Evaluations*

To evaluate the performance of the optimal SDQ that is found via DP, we assume an $N$-ASK source that is disturbed by AWGN and the resulting continuous output channel is uniformly pre-quantized into $N_\mathsf{y}$ clusters. We note that using a finely QAWGN channel in the design is not limiting the performance, since the pre-quantization loss can be made arbitrary small by using a refined partition [3]. We compare the performance of the DP approach with the KL-means [25] and greedy combining [26] for 4-, 8- and 16-ASK on the QAWGN channel with $N_\mathsf{y} = 128$. The complexity of the KL-means algorithm is $\mathcal{O}(N N_\mathsf{y}^2 i_\mathsf{max} N_\mathsf{init})$, where $i_\mathsf{max}$ is the maximum number of iterations and $N_\mathsf{init}$ is the number of randomly initialized executions. In contrast to that, greedy combining has complexity $\mathcal{O}(N N_\mathsf{y}^2 (N_\mathsf{y} - N_\mathsf{z}))$.

Fig. 6 shows the resulting KL loss $L(Q) = I(\mathsf{x};\mathsf{y}) - I(\mathsf{x};\mathsf{z})$ if the DMC is quantized into $N_\mathsf{z} = 8, ..., 32$ clusters. The KL-means algorithm is executed $N_\mathsf{init} = 10^5$ times and the best result is stored. The optimal SDQ that is found via the DP approach is denoted as DP-Opt.

The DP approach outperforms the KL-means algorithm and greedy combining in all cases. We also observe an increasing gain of the DP approach if the input cardinality

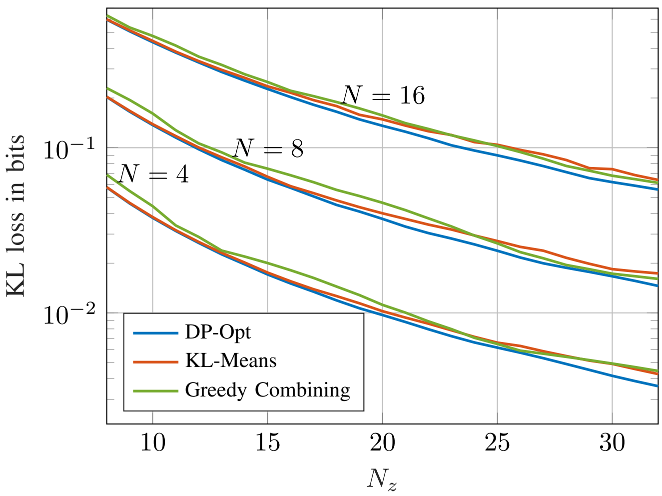


Fig. 6: KL loss for DP-Opt, KL-means and greedy combining for different number of clusters $N_\mathsf{z}$. QAWGN channel with $\sigma_\mathsf{n}^2 = 1$ and uniformly distributed $N$-ASK input.

$N$ is increasing. Furthermore, the complexity of the DP approach is $\mathcal{O}(NN_\mathsf{z}(N_\mathsf{y} - N_\mathsf{z}))$ since the loss function for the QAWGN channel fulfills the quadrangle inequality [21]. Hence, the DP approach achieves best performance and has significantly lower complexity compared to KL-means and greedy combining.

## VI. Conclusion

The problem of mutual information maximizing quantizer design for real valued input and output communication channels of a physical transmission was investigated. We exploited the equivalent formulation as a clustering problem in the $N-1$ dimensional probability simplex. We observed that for $N$-ASK input with AWGN transmission all channel output values are on a *smooth curve* in this simplex since they follow a natural ordering of likelihood ratios. For sufficient fine pre-quantization, the global optimal quantizer is found via Dynamic Programming (DP). Furthermore, the DP approach outperforms quantizer design algorithm such as KL-means and greedy combining in terms of complexity and performance.

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