# Instantaneous Bandwidth Estimation for Efficient Sampling of Electrocardiograms

Christopher Willuweit, Johannes Königs, Carsten Bockelmann, Armin Dekorsy
Dept. of Communications Engineering
University of Bremen, Bremen, Germany
{willuweit, koenigs, bockelmann, dekorsy} @ant.uni-bremen.de

***Abstract*—The Nyquist-Shannon sampling theorem states that bandlimited signals can be perfectly reconstructed from samples taken at a fixed rate. Signals with varying spectral content are not considered, which leads to an unnecessarily high number of samples in signal intervals with narrowband content. An extension of the Nyquist-Shannon theorem enables the definition of variable bandwidth signals through nonlinear time axis distortion. This technique, known as time warping, enables variable-rate sampling based on instantaneous bandwidth, resulting in sample numbers proportional to the average bandwidth rather than the maximum bandwidth as in classical sampling. In practice, however, the instantaneous bandwidth of a signal is unknown, except for a few analytically determinable exceptions. In this paper, we introduce a novel spectrogram-based algorithm for estimating the instantaneous bandwidth of classically sampled signals, allowing to project them to variable bandwidth signals. We examine the tradeoff between sample reduction and reconstruction accuracy of electrocardiograms and compare the results to classical downsampling.**

***Index Terms*—Instantaneous Bandwidth, Time-warping, Nonuniform sampling, Electrocardiogram (ECG)**

## I. Introduction

The Nyquist-Shannon sampling theorem is the foundation for digital signal processing of analog signals. It states that bandlimited signals can be perfectly reconstructed from equidistant samples. The sampling frequency is dependent on the maximum frequency component of the signal and is kept fixed in conventional signal processing systems. However, the frequency content of many practical signals changes significantly over time. Some notable examples are speech signals [1] [2], frequency modulated (FM) signals [3], and electrocardiograms [4] [5]. The Nyquist-Shannon sampling theorem does not account for periods with narrowband spectral components where the sampling frequency is too high, resulting in more samples than necessary.

There exist several approaches to sample signals more efficiently, including compressive sensing [6], finite rate of innovation [7], and event-based sampling [8]. Another method, called *time-warping* was introduced in [3] by Clark, Palmer, and Lawrence. Their proposal extends the classical sampling theorem for nonuniform sampling distributions by introducing a nonlinear distortion of the time-axis of classically bandlimited signals. They use the resulting instantaneous sampling rate to define a measure of *instantaneous bandwidth* $B(t)$ of the warped signals. Therefore, this framework serves not only as a sampling scheme but also as a signal model that accommodates structural characteristics beyond constant band limitation. We call signals modeled in this way *variable bandwidth (VBW)* signals.

Event-based samplers adjust their sampling rate to the instantaneous bandwidth of VBW signals [8], which, when combined with event-based communication, promises significantly enhanced energy efficiency in wireless communication [9]. To evaluate potential sample reductions, practical signals must be modeled as VBW signals, which requires an estimate of instantaneous bandwidth. Typically, practical signals are uniformly sampled under the assumption of classical band limitation. By transforming these signals into VBW signals, we can estimate potential sample savings, especially in event-based sampling approaches.

An existing method to project signals into VBW signals employs an iterative time-warping algorithm [10] utilizing an undisclosed procedure involving the short-time Fourier transform (STFT). In our work, we demonstrate how the STFT, specifically the spectrogram, can be used to estimate the instantaneous bandwidth of classically sampled signals. We introduce our developed algorithm and employ it to project classically sampled signals into VBW signals using the time-warping technique from [3]. To illustrate the benefits of this representation, we apply it to electrocardiogram (ECG) signals, achieving a reduction in the total number of samples compared to classical sampling. We also investigate the trade-off between sample reduction and reconstruction error using classical downsampling for comparison.

## II. Time-warping

First, we will revisit the Nyquist-Shannon sampling theorem. Afterwards, we introduce the time-warping extension of [3] which is used to define a measure of instantaneous bandwidth. The Nyquist-Shannon sampling theorem states that, if a signal $x(\tau)$ is bandlimited to the bandwidth $B_0$,

$$\forall |f| > B_0 : \quad X(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau = 0, \quad (1)$$

it can be reconstructed by its equidistant samples $x(\frac{n}{2B_0}), n \in \mathbb{N}$ with a constant sampling rate $f_0 = 2B_0$ using the Whittaker-

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Shannon interpolation formula:

$$x(\tau) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2B_0}\right) \cdot \mathrm{sinc}(2B_0\tau - n). \tag{2}$$

Clark, Palmer and Lawrence [3] introduced an extension of this theorem by defining a signal $y(t) = x(\gamma(t))$ as a bandlimited signal $x(\tau)$ which is transformed by a strictly monotonically increasing *time-warping* function $\tau = \gamma(t)$. For a simpler understanding of this concept and without loss of generality we set the bandwidth of $x(\tau)$ to be $B_0 = \frac{1}{2}$ for this explanation. Substituting the time-warping function $\gamma(t)$ into (2) we get

$$y(t) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{sinc}(\gamma(t) - n). \tag{3}$$

Since $\gamma(t)$ is bijective, we can formulate the samples in the $\tau$-domain $x(n)$ as samples in the $t$-domain $y(\gamma^{-1}(n))$ and get the interpolation formula

$$y(t) = \sum_{n=-\infty}^{\infty} y(\gamma^{-1}(n))\mathrm{sinc}(\gamma(t) - n). \tag{4}$$

We call such signals $y(t)$ variable bandwidth (VBW) signals. They are represented by the samples $y(t_n)$ with

$$t_n = \gamma^{-1}(n) \tag{5}$$

and the time-warping function $\gamma(t)$. Thus, the sample times are the integer crossings of $\gamma(t)$. The instantaneous sampling rate of the samples can be formulated as the derivative of $\gamma(t)$ [3]

$$f_s(t) = \frac{\delta\gamma(t)}{\delta t}. \tag{6}$$

We assume here, as in classical sampling, that the bandwidth is twice the sampling frequency, except that here both the sampling frequency and bandwidth are functions of time $t$:

$$B(t) := \frac{1}{2} \cdot \frac{\delta\gamma(t)}{\delta t}. \tag{7}$$

By integration, we can also find $\gamma(t)$ when $B(t)$ is given:

$$\gamma(t) = 2\int B(t)dt. \tag{8}$$

In the subsequent sections of this paper, we will represent VBW signals with $B(t)$ and the non-uniform samples $y(t_n)$.

The number of samples in a given time interval $t \in [0, T]$ for Shannon-Nyquist sampling is $2 \cdot B_0 \cdot T$. The number of samples for VBW signals in the same interval is $\gamma(T) - \gamma(0)$ which corresponds to the average instantaneous bandwidth $\overline{B} = \frac{2}{T}\int_0^T B(t)dt$ in the time interval. As long as the average bandwidth $\overline{B}$ of a VBW signal is smaller than the constant bandwidth $B_0$ of a bandlimited signal, VBW signals require fewer samples than classically bandlimited signals.

In the next section, we present a method for determining $B(t)$ of uniformly sampled bandlimited signals $x(t)$ with any given $B_0$. This provides a means of determining $\gamma$ and subsequently the non-uniform sampling times $t_n$. By resampling $x(t)$ using (2) at $t_n$ we can project the bandlimited signal into a VBW signal.

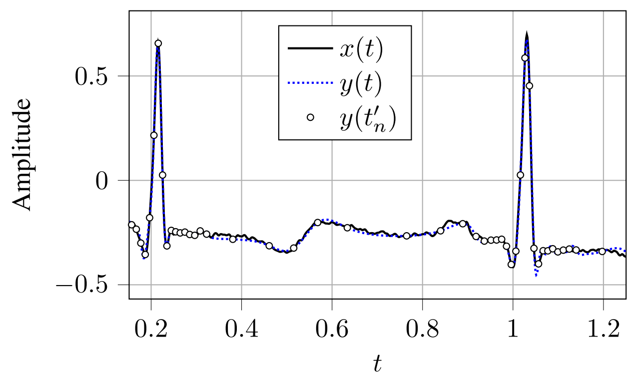


Fig. 1. Example ECG signal $x(t)$, time-warped samples $y(t'_n)$, and reconstructed VBW signal $y(t)$.

## III. Instantaneous bandwidth estimation and resampling

Our algorithm is based on the spectrogram, a representation of local signal energy over time and frequency. We will give an intuitive view of how we get from the spectrogram to an estimation of instantaneous bandwidth before we define it mathematically. Using an example ECG signal sampled at 360 Hz with two heartbeats (shown as a solid black graph in Fig. 1), we find that its bandlimited representation contains spectral content up to 180 Hz. Fig. 2 depicts the spectrogram of this signal (only the colored part), showing only the section up to 100 Hz where most of the signal energy is located. The algorithm described below estimates the instantaneous bandwidth (shown as a solid black graph in Fig. 2) based on an energy criterion. We then project the bandlimited signal into a VBW signal, causing two effects: First, the VBW signal differs from the original since the energy components above the estimated instantaneous bandwidth are removed. Second, the VBW signal can be represented with fewer samples as the sampling rate is adapted to the time-varying spectral content instead of maintaining a fixed rate of $2B_0$. Fig. 1 shows the nonuniform samples and the reconstructed VBW signal, highlighting the high sample concentrations in areas of high instantaneous bandwidth.

The bandwidth estimation is based on the spectrogram of real signals sampled uniformly at $f_0$. We denote these time-discrete signals as $x[n]$ where $x[n] = x(\frac{n}{f_0})$. The signals have a length of $N$ samples, hence $n \in [0, N-1]$. The spectrogram for time-discrete signals is defined as

$$S[m,k] = \frac{1}{N_w}\left|\sum_{n=0}^{N_w-1} x[n]w[n-m]e^{-j\frac{2\pi}{N_w}nk}\right|^2, \tag{9}$$

where

$$m \in [0, M-1], \qquad k \in [0, K-1], \tag{10}$$

$$M = N - N_w, \qquad K = \left\lceil \frac{N_w}{2} \right\rceil. \tag{11}$$

$w[n]$ is an energy normalized time-discrete window function which is nonzero only for $n \in [0, N_w - 1]$, where $N_w \leq N$.

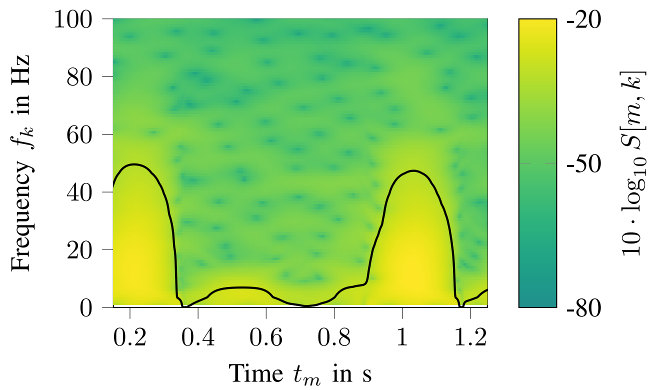


Fig. 2. Spectrogram of example ECG signal $x(t)$ (logarithmic, colored) and instantaneous bandwidth estimate $B(t)$ (solid black).

Thus, the spectrogram performs discrete Fourier transformations on windowed segments of length $N_w$ of the signal $x[n]$.

$S[m,k]$ is an estimation of the power spectral density of the signal $x[n]$ at time $t_m$ and frequency $f_k$:

$$t_m = \frac{m}{f_0} + \frac{N_w}{2f_0}, \qquad f_k = \frac{f_0}{N_w} \cdot k. \tag{12}$$

For further details on the spectrogram we refer to [11].

To find $B(t)$, we now sum up all energies cumulatively over the frequency index $k$ per time segment $m$ of the spectrogram $S[m,k]$. This results in a function

$$S^+[m,k] = \sum_{k'=0}^{k} S[m,k'], \tag{13}$$

showing us how much energy is contained between the frequency zero and the frequency $f_k$ at time $t_m$. Our algorithm relies on an energy threshold, so we need a continuous cumulative energy distribution $s_m^+(f)$ for each time segment $m$ which we calculate by piecewise linear interpolation of $S^+[m,k]$. As the frequencies $f_k$ are equidistant, we can use the convolution with the triangular function to describe this interpolation:

$$s_m^+(f) = \sum_{k=0}^{K-1} S^+[m,k] \cdot \text{tri}\left(\frac{N_w}{f_0}(f - f_k)\right). \tag{14}$$

If we now had the task to find an instantaneous bandwidth that preserves all the energy contained in the signal, we would end up at the originally assumed constant bandwidth $B(t) = B_0$. Thus, in order to save samples by lowering the instantaneous bandwidth from its original constant value of $B_0$, we must necessarily discard signal energy. We formulate the discarded signal energy $e_\text{d}$ as a fraction $q$ of the total signal energy $e_x = \sum_{n=0}^{N-1} |x[n]|^2$:

$$e_\text{d} = q e_x, \tag{15}$$

where $q$ denotes the ratio of signal energy discarded.

Next, we distribute $e_\text{d}$ evenly over all time segments of the spectrogram. Since a spectrogram segment has a length of $N_w$ and the window $w[n]$ is chosen to be energy conserving, the energy to be discarded per segment $e_\text{d,seg}$ is

$$e_\text{d,seg} = e_\text{d} \frac{N_w}{N}. \tag{16}$$

The energy conserved per segment is dependent on $m$ and can be formulated as

$$e_{\text{thr},m} = S^+[m, K-1] - e_\text{d,seg}, \tag{17}$$

where $S^+[m, K-1]$ is the total energy of segment $m$. To find the estimated instantaneous bandwidth per segment, we solve $e_{\text{thr},m} = s_m^+(f)$. As long as $e_{\text{thr},m} \geq s_m^+(0)$, the solution is guaranteed to exist, because $S^+[m,k]$ is purely positive and thus $s_m^+(f)$ is monotonically increasing. Nonexisting solutions are set to zero:

$$\tilde{B}(t_m) = \begin{cases} 0, & \text{if } e_{\text{thr},m} < s_m^+(0) \\ f : s_m^+(f) = e_{\text{thr},m}, & \text{else.} \end{cases} \tag{18}$$

The solution might not be unique, in which case we select the smallest solution. This way we get a preliminary estimation $\tilde{B}(t_m)$ of the instantaneous bandwidth at the times $t_m$.

The time-warping framework requires purely positive $B(t)$ because the corresponding $\gamma(t)$ has to be strictly monotonically increasing. Hence, we define a free parameter, the minimal bandwidth $B_\text{min}$ and clip $\tilde{B}(t)$ to get the final estimate of the instantaneous bandwidth:

$$B(t_m) = \max(B_\text{min}, \tilde{B}(t_m)). \tag{19}$$

To get from the estimates at $t_m$ to a continuous function we again use piecewise linear interpolation:

$$B(t) = \sum_{m=0}^{M-1} B(t_m) \cdot \text{tri}\left(f_0(t - t_m)\right). \tag{20}$$

We can now compute $\gamma(t)$ using (8), and as $B(t)$ is piecewise linear, we can determine it analytically. Subsequently, we determine the sampling times $t_n$ using (5). We then resample the original signal $x(t)$ at $t_n$ using $x[n]$ and the sinc-interpolation (2). The resulting number of nonuniform samples is ensured to be less than or equal to the original number of samples $N$ because $B(t)$ cannot exceed the original bandwidth $B_0$. To determine the balance between saved samples and reconstruction error, we will present numerical findings in Section V.

## IV. Parametrization for ECG Signals

### A. Preprocessing

To show the performance of the proposed algorithm, we use recorded ECG data from the MIT-BIH Arrhythmia Database [12] from PhysioNet [13]. The used recordings are from the normal electric activity within the heart. The data has a sampling frequency of $f_s = 360\,\text{Hz}$, which is, according to [14], highly oversampled. To remove unwanted noise especially in the high frequency domain we low-pass filter[1] the data with

[1] Third-order Butterworth

a cutoff frequency of 100 Hz. Since artifacts from the power grid are visible in the raw data we additionally apply a notch filter[2] with a center frequency of 60 Hz. Afterwards, we divide the data into sections of 512 samples and subtract the mean from the signal before applying the spectrogram, as is usual for ECG signals [15].

### B. Bandwidth estimation

The parameter $N_w$ and the selection of the window function $w[n]$ jointly determine the spectrogram's time and frequency resolution and are subject to the Gabor limit [16]. We refer to [11] for the selection of the window function. Here we choose the Hann window as it performs the best among the options examined:

$$w[n] = \begin{cases} \frac{1}{2}\left(1 - \cos\left(\frac{2\pi n}{N_w - 1}\right)\right), & \text{for } n \in [0, N_w - 1] \\ 0, & \text{else.} \end{cases} \quad (21)$$

The length of the segments was also optimized numerically, and we obtained the best results with $N_w = 100$. Several spectrogram variants, such as zero-padding of segments to improve frequency resolution or evaluating more than a subset of all possible $m$ (see (10)), were not explored in this study because of their minor impact on the results. For the minimum instantaneous bandwidth a value of $B_{\text{min}} = 0.1$ Hz was chosen.

## V. Numerical Evaluation

### A. Average sampling rate and reconstruction error

We want to analyze the tradeoff between the number of samples for signal representation and reconstruction accuracy. We analyze the sample count using the average sampling rate (ASR). After determining the instantaneous bandwidth $B(t)$ we obtain the ASR from the integer crossings of $\gamma(t)$ during the signal duration $T$

$$\text{ASR} = \frac{\gamma(T) - \gamma(0)}{T} \quad (22)$$

as described in Section II.

For the reconstruction error we use the normalized mean squared error (NMSE) of the VBW representation $y(t)$ with respect to the original bandlimited representation $x(t)$. We evaluate both signal representations at the original uniform sampling rate of $f_0 = 360$ Hz for the error calculation:

$$\text{NMSE} = \frac{\sum_{n=1}^{N} \left| x(\frac{n}{f_0}) - y(\frac{n}{f_0}) \right|^2}{\sum_{n=1}^{N} \left| x(\frac{n}{f_0}) \right|^2}. \quad (23)$$

### B. Alternative approach for sample reduction

To evaluate the tradeoff between ASR and NMSE of our presented approach, we compare it with the reconstruction error of classical downsampling, a simple alternative method for sample reduction. Here, we first low-pass filter[3] the original signal $x(t)$ for anti-aliasing and then resampled at a uniform rate $r \cdot f_0$, where the downsampling factor $r < 1$. The resulting

[2]Second-order, quality factor $Q = 30$.

[3]Third-order Butterworth, cutoff at twice the reduced sampling rate

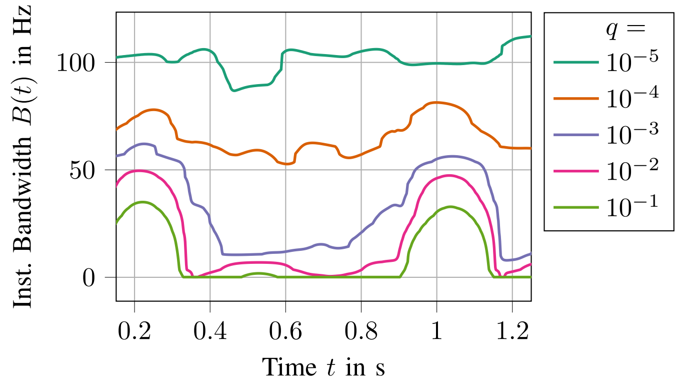


Fig. 3. Resulting instantaneous bandwidths $B(t)$ for different energy ratios $q$ of the example ECG signal.

NMSE is calculated according to (23) with $y(t)$ replaced by the downsampled signal. By varying the downsampling factor, we can obtain different ASRs $r \cdot f_0$.

Apart from the anti-aliasing filter, this approach is equivalent to using the VBW representation of a signal using a constant instantaneous bandwidth $B(t) = r \cdot f_0$.

### C. Results

By varying the energy ratios $q$, we can achieve VBW representations $y(t)$ with different ASRs. The choice of $q$ leads to different instantaneous bandwidths as is shown in Fig. 3 for the example ECG signal. The case $q = 10^{-1}$ in particular shows the limitation of $B(t)$ to $B_{min} = 0.1$. Furthermore, the instantaneous bandwidth increases with decreasing $q$ and approaches a constant curve.

For the final results we analyzed 500 different ECG signals as described in Section IV-A. We calculated the instantaneous bandwidths for each signal with 100 different logarithmically spaced energy ratios $q$ between $2 \cdot 10^{-1}$ and $10^{-5}$. Then, we projected the signals into their VBW representation and calculated the NMSE to the original representation. In case of classical downsampling, we selected 100 different uniformly spaced downsampling factors $r$ between 0.03 and 0.3. Fig. 4 shows the resulting NMSE over the ASR for our approach (solid line) and the classical downsampling (dashed line).

Our approach demonstrates its superiority for ASRs below approximately 140 Hz. In certain cases, achieving an equivalent NMSE to downsampling requires only half the number of samples. For ASRs exceeding 140 Hz, both curves roughly align. This can be attributed to the behavior observed for smaller $q$, where the instantaneous bandwidth approaches a constant value, akin to classical downsampling performance. However, it is worth noting that the downsampling procedure incorporates an anti-aliasing lowpass filter, while our approach does not. Consequently, an additional error stemming from aliasing emerges. We believe this effect accounts for the worse NMSEs of our approach compared to downsampling for ASRs exceeding 140 Hz.

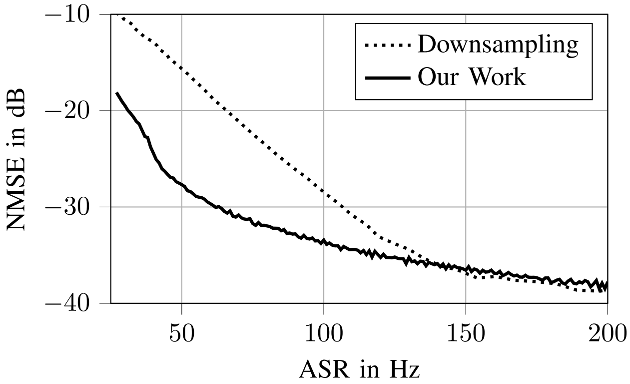


Fig. 4. Numerical analysis of 500 ECG signals showing the relation between NMSE and the ASR for classical downsampling and our approach.

## VI. Conclusion and Outlook

In this paper, we presented a novel algorithm to efficiently resample classically bandlimited signals according to a variable bandwidth (VBW) framework. Our algorithm is based on the spectrogram in combination with energy-thresholding and offers a free parameter, $q$ that can be used to balance the number of samples and the reconstruction error. The results show a clear superiority of the VBW signal representation compared to classical downsampling.

Moreover, the presented algorithm allows modeling practical signals as VBW signals. This class of signals provides an interesting theoretical framework for describing signal structure and, in combination with event-based samplers, promises an efficient communication of structured signals.

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