# Effectiveness of Time-Warping in Signal Reconstruction from Nonuniformly Distributed Level-Crossing Samples

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***Abstract*—The reconstruction of bandlimited signals from nonuniform samples remains a key challenge in modern signal processing, particularly for variable bandwidth (VBW) signals. LC sampling, a practical event-based sampling technique, adapts the sampling rate to the instantaneous bandwidth of a signal, but the inherent irregular sample spacing complicates signal reconstruction. In this paper, we analyze the effectiveness of time-warping in improving the sample spacing of level-crossing (LC) samples for VBW signals and its impact on minimum energy (ME) reconstruction. By time-warping LC samples, we demonstrate a reduction in sample spacing variability, leading to a more stable reconstruction process. We evaluate the sample spacing distribution before and after time-warping and compare the reconstruction performance of ME reconstruction with and without time-warping. Numerical results show that time-warping significantly enhances reconstruction accuracy, reducing the numerical instability of the ME method. This study highlights the potential of time-warping as an effective preprocessing step for LC sampled VBW signals.**

***Index Terms*—Event-Based Sampling, Instantaneous Bandwidth, Time-warping**

## I. Introduction

The frequency content of many practical signals we want to acquire changes significantly over time. This is especially the case for signals like speech signals [1] [2], frequency modulated (FM) signals [3], and electrocardiograms [4] [5].

Since the Nyquist-Shannon sampling theorem states that signals can be perfectly reconstructed from equidistant samples, we sample most signals dependent on the maximum frequency component of the signal and keep this rate fixed in conventional signal processing systems. This does not account for periods with narrowband spectral components, resulting in an unnecessarily high sampling rate and thus excessive signal sampling. There exist several approaches to sample signals more efficiently, including compressive sensing [6], finite rate of innovation [7], and event-based sampling [8].

A different approach, called *time-warping* extends the classical sampling theorem by adding support for *variable bandwidth (VBW)* signals [3]. VBW signals are fully characterized by a sequence of nonuniform samples taken according to an instantaneous, rather than a globally defined, signal bandwidth. A distortion function derived from the instantaneous signal bandwidth warps the time-axis of the signal such that the nonuniform samples become uniformly distributed. This allows simple signal recovery based on the Nyquist-Shannon interpolation method. Although an interesting concept, its sampling process requires a priori knowledge about the signals instantaneous bandwidth and is therefore impractical.

A far more convenient approach to nonuniform sampling, which does not require a priori knowledge of a signals instantaneous bandwidth, is the event-based sampling technique level-crossing (LC) sampling. Here, a signal is sampled whenever it crosses a predefined amplitude threshold. This leads to a highly nonuniform sampling grid with potentially large distances between samples when the signal activity is low. It has been shown that the samples are taken at a variable rate in relation to the bandwidth or, in the case of a VBW signal, the instantaneous bandwidth of the signal [9]. Consequently, signal recovery is challenging and complex reconstruction techniques are required. Existing methods include spline-based filtering [10], projections onto convex sets [11] and a technique that combines time-warping with minimum energy (ME) reconstruction [12].

In this paper we want to investigate the effect of time-warping on the spacing of the nonuniform LC sample grid for VBW signals and demonstrate that it can enhance the performance of theME reconstruction. To this end, we will first introduce ME reconstruction, the theory behind time-warping, and then examine the relationship between the bandwidth of a signal and its LC sample distribution. Then, we will discuss the effect of time-warping on the LC sample grid for an example signal and subsequently analyze the distribution of the sample spacing numerically. Finally, we will compare the ME reconstruction from LC samples with and without prior time-warping.

## II. Nyquist-Shannon Sampling Theorem

A signal $y(t)$ that is bandlimited to the bandwidth $B_0$

$$\forall |f| > B_0: \quad Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt = 0, \qquad (1)$$

can be fully recovered from uniformly spaced samples $y(\frac{m}{2B_0}), m \in \mathbb{M}$ taken at a constant rate of $f_0 = 2B_0$.

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For recovery, we most commonly use the Whittaker-Shannon interpolation formula:

$$y(t) = \sum_{m=-\infty}^{\infty} y\left(\frac{m}{2B_0}\right) \cdot \text{sinc}(2B_0 t - m). \tag{2}$$

## III. Minimum Energy Reconstruction

If we now sample $y(t)$ irregularly, resulting in $K$ nonuniform samples $y(t_k)$, then (2) will not provide an error free reconstruction. Instead, we can use ME reconstruction [13], which by finding a bandlimited $\hat{y}(t)$ minimizes the least squares error:

$$e(\boldsymbol{c}) = \min_{\boldsymbol{c}} \|y(t) - \hat{y}(t)\|^2. \tag{3}$$

This is accomplished by selecting a finite number of reconstruction functions:

$$\hat{y}(t) = \sum_{n=1}^{N} \boldsymbol{c}[n] g_n(t), \tag{4}$$

where $\boldsymbol{c}$ can be found from the observations $y(t_k)$ by solving:

$$y(t_k) = \sum_{n=1}^{N} \boldsymbol{c}[n] \text{sinc}(2B(t_k - t_n)) \tag{5}$$

for all $y(t_k)$. This leads to a linear equation system to solve for all $K$ samples:

$$\boldsymbol{y} = \boldsymbol{G}\boldsymbol{c}. \tag{6}$$

$\boldsymbol{G}$ is a $K \times N$ matrix containing the reconstruction kernels

$$\boldsymbol{G}[k, n] = \text{sinc}(2B(t_k - t_n)) \tag{7}$$

and $\boldsymbol{y} \in \mathbb{R}^{K \times 1}$ are our observations $\boldsymbol{y}[k] = y(t_k)$.

When the sample times $t_k$ are highly nonuniformly distributed, the matrix $\boldsymbol{G}$ is ill-conditioned [14]. Inverting $\boldsymbol{G}$ can thus lead to large numerical errors in the reconstruction. To compensate, we apply Miller regularization [12].

## IV. Time-warping

Clark et al. [3] introduced an extension of the sampling theorem for nonuniform sample sequences of VBW signals called time-warping. We again define a signal $x(\tau)$ that is bandlimited, now to $B = \frac{1}{2}$ in a bandwidth-normalized *$\tau$-domain*. In the $\tau$-domain, we can describe the signal $x(\tau)$ by uniformly spaced samples $x(n)$. We define a function $\gamma(t)$ which we require to be invertible and strictly monotonically increasing. For a signal $y(t) = x(\tau)$ with $\tau = \gamma(t)$ the resulting samples $y(\gamma^{-1}(n))$ do not necessarily lie on a uniform grid, instead their spacing is defined by the shape of $\gamma(t)$. The time axis of our signal $y(t)$ ($t$-domain) and of our signal $x(\tau)$ ($\tau$-domain) are thus related by $\tau = \gamma(t)$. If $\gamma(t)$ is not a linear function, then the samples $y(\gamma^{-1}(n))$ are nonuniformly distributed over the $t$-domain as shown in Fig. 1.

Substituting the time-warping function $\gamma(t)$ into the Whittaker-Shannon interpolation formula (2) gives:

$$y(t) = x(\gamma(t)) = \sum_{n=-\infty}^{\infty} x(n) \cdot \text{sinc}(\gamma(t) - n). \tag{8}$$

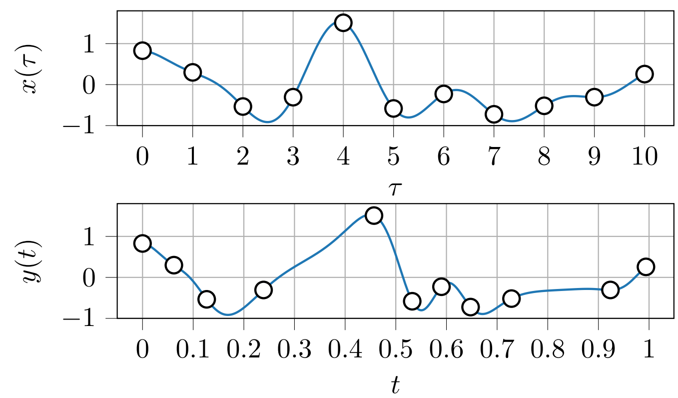


Fig. 1. The signal $x(\tau)$ with its uniform samples $x(n)$ in the $\tau$-domain and its representation $y(t)$ in the $t$-domain with nonuniform samples $y(t_n)$.

We can express the $\tau$-domain samples $x(n)$ as the samples $y(\gamma^{-1}(n))$, since $\gamma(t)$ is a bijective function, to obtain the time-warped interpolation formula:

$$y(t) = \sum_{n=-\infty}^{\infty} y(\gamma^{-1}(n)) \cdot \text{sinc}(\gamma(t) - n). \tag{9}$$

Thus, the signal $y(t)$ can be fully recovered from nonuniform samples $y(t_n)$ given by the inverse time-warping function:

$$t_n = \gamma^{-1}(n). \tag{10}$$

Since the sampling times are the integer crossings of $\gamma(t)$, the derivative of $\gamma(t)$ describes the instantaneous sampling rate:

$$f_s(t) = \frac{\delta\gamma(t)}{\delta t} \tag{11}$$

from which we can define bandwidth as a function of $t$:

$$B(t) := \frac{1}{2} \cdot \frac{\delta\gamma(t)}{\delta t}. \tag{12}$$

Likewise, if the instantaneous bandwidth $B(t)$ of the signal, is known, we can find the time-warping function $\gamma(t)$ by integration:

$$\gamma(t) = 2 \cdot \int B(t)dt. \tag{13}$$

In summary, time-warping allows signal recovery from nonuniformly samples, if $B(t)$ is known and the nonuniform samples are taken at the exact times we obtain from $\gamma(t)$.

## V. Level-crossing Sampling

A more practical approach to nonuniform sampling in relation to the instantaneous bandwidth of a signal is level-crossing (LC) sampling. A LC sampler samples a signal $x(t)$ whenever the signal amplitude crosses a certain threshold. For a sampler with a single level $L$, crossing its level would yield the sample $x(t_k) = L$.

In the following, we define $x(t)$ as a stationary Gaussian process. The average number of crossings in the interval $[0, T]$

$$K_T(L) = \#\{t \in [0, T] : x(t_k) = L\} \tag{14}$$

can then be described by the Rice formula [15]:

$$\mathbb{E}\left[K_T(L)\right] = \frac{T}{\pi}\sqrt{\frac{\lambda_2^x}{\lambda_0^x}} \cdot e^{\frac{-L^2}{2\lambda_0^x}}. \tag{15}$$

Here, $\lambda_0^x = \mathrm{var}\left[x(t)\right]$ and $\lambda_2^x = \mathrm{var}\left[x^{(1)}(t)\right]$ are the first and second spectral moments of $x(t)$ and its first derivative $x^{(1)}(t)$. There is a direct link between the process bandwidth and the expected number of transitions, since the mean process bandwidth $\bar{B}$ of $x(t)$ can be found from its spectral moments:

$$\bar{B} = \sqrt{\frac{\lambda_2^x}{\lambda_0^x}}. \tag{16}$$

Applying the time-warping method, we can model a non-stationary Gaussian process $y(t)$ as a VBW signal by time-warping a stationary process $y(t) = x(\gamma(t))$. Rzepka et al. showed in [9] that a similar relationship between the instantaneous bandwidth $B(t)$ of $y(t)$ and the average number of LC samples can be established:

$$\mathbb{E}\left[K_T(L)\right] = \frac{1}{\pi} e^{\frac{-L^2}{2\lambda_0^y}} \int_0^T \sqrt{\frac{\lambda_2^y}{\lambda_0^y}} 2\pi B(t) dt. \tag{17}$$

## VI. Application of time-warping to Signal Recovery from LC samples

In the following section we want to analyze the time-warping projection and its effects on the ME reconstruction of a VBW signal from its nonuniform LC samples. We will first project the LC samples of an example signal onto the $\tau$-domain and analyze the resulting sample distances. Then, we will look at the reconstruction performance of the combined time-warping and ME reconstruction approach as described by Rzepka et al. [12].

### A. Example VBW Signal

For our test signal, we define a VBW signal $y(t)$ with a known instantaneous bandwidth $B(t)$. We limit our observation interval to $t \in [0, 1]\,\mathrm{s}$ and require that $B(t)$ is a bandlimited function with a bandwidth $\Upsilon = 2\,\mathrm{Hz}$. We construct $B(t)$ from random amplitudes $B_m \in \mathcal{U}\left(0.01\,\mathrm{Hz}, 10\,\mathrm{Hz}\right)$ using (2). From $B(t)$ we can find a time-warping function $\gamma(t)$ by integration, and its resulting nonuniform samples by solving (10). Then, by randomly selecting $N$ amplitudes $y(t_n)$, we can construct our VBW signal $y(t)$ by using (9). We choose $y_n \in \mathcal{N}(0, 1)$ from the normal distribution such that $-4 \leq y(t) \leq 4$. Fig. 2 shows the random instantaneous bandwidth $B(t)$, the signal $y(t)$ and its $N$ nonuniform samples $y(t_n)$.

We then sample $y(t)$ with a LC sampler with $N_L = 15$ equally distributed levels between $[-4, 4]$, resulting in the samples $y(t_k)$. During times when $B(t)$ is large, more nonuniform samples $y(t_n)$ are required to describe $y(t)$. Additionally, more LC samples $y(t_k)$ are generated at those times due to the higher signal activity. The opposite is true when $B(t)$ is low.

Since the LC samples $y(t_k)$ do not overlap with the nonuniform samples $y(t_n)$, thus not forming a uniform sampling grid

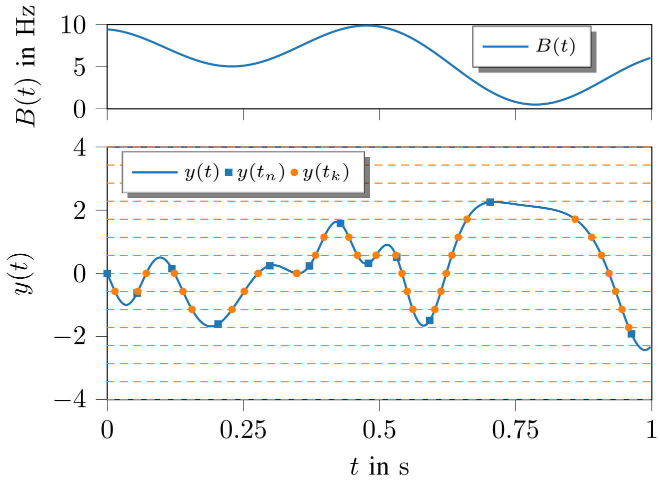


Fig. 2. An example instantaneous bandwidth $B(t)$, the resulting VBW signal $y(t)$ and its nonuniform samples $y(t_n)$, sampled with a LC sampler, resulting in the samples $y(t_k)$.

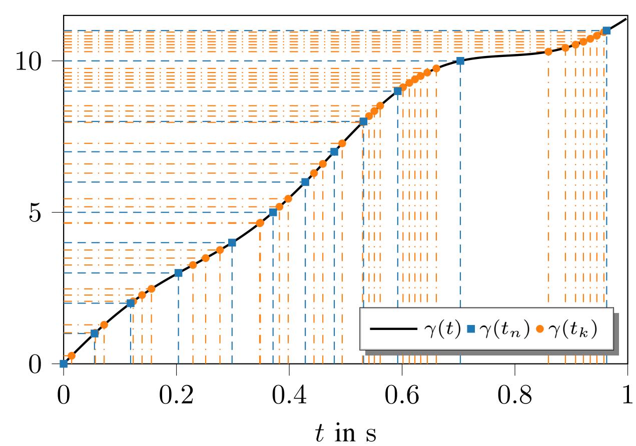


Fig. 3. Time-warping function $\gamma(t)$ of Fig. 2 and the nonuniform sample times $t_n$ and the LC sample times $t_k$ in $t$-and $\tau$-domain.

in the $\tau$-domain, a direct reconstruction with the time-warped interpolation formula (9) is not feasible.

Fig. 3 shows the projection of the LC samples $y(t_k)$ to the $\tau$-domain using $\gamma(t)$. We can observe that for most nonuniform sample times $t_n$, which form a uniform sampling grid in the $\tau$-domain, adjacent LC samples $t_k$ occur. Although there are still large gaps in the $\tau$-domain LC sample grid, the clustered LC samples are now close to evenly spaced with different rates per cluster.

### B. Sample Time Distance Distribution

In the following section we want to extend the previous observations by examining the distributions of the sample spacing of LC samples in the $t$ and $\tau$ domains. To this end we construct 1000 VBW signals. To generate the instantaneous bandwidths we choose $B_m \in \mathcal{U}\left(5\,\mathrm{Hz}, 100\,\mathrm{Hz}\right)$ for greater variability in sample rates and sample them with LC samplers

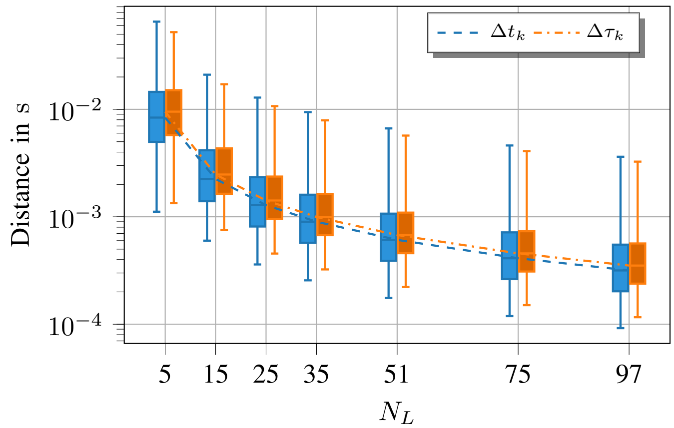


Fig. 4. Sample time distance distribution in $t$-and $\tau$-domain.

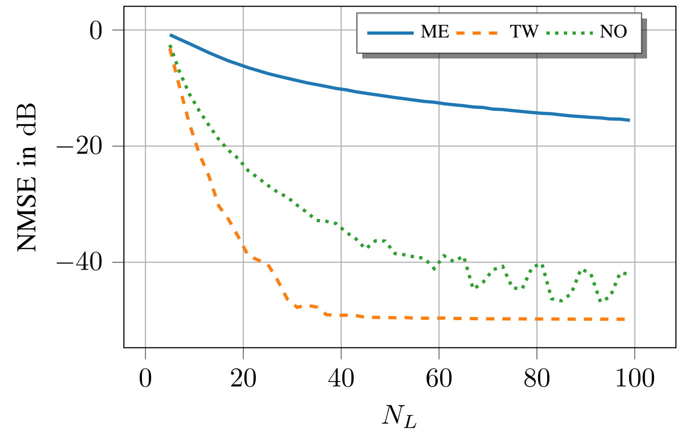


Fig. 5. Reconstruction comparison between ME, ME in $\tau$-domain (TW) and ME in $\tau$-domain from selected samples only (NO).

with varying number of levels $N_L \in \{3, 5, \dots, 95, 97\}$, resulting in the samples $y(t_k)$.

We then calculate the distances between all LC samples:

$$\Delta t_k = t_{k+1} - t_k, \quad k \in \{0, \dots, K-1\} \tag{18}$$

for all signals and sampler configurations and repeat the process for the time-warped samples:

$$\Delta \tau_k = \frac{(\tau_{k+1} - \tau_k)}{\gamma(t_{\max}) - \gamma(t_{\min})}, \quad k \in \{0, \dots, K-1\}. \tag{19}$$

The resulting distributions of the sample distances over the number of levels $N_L$ shown in Fig. 4 are all positively skewed. We can observe that projection to the $\tau$-domain reduces the interquartile range of the box plots. The whiskers have been configured to show the $1\%$ and $99\%$ minimum and maximum to better represent the spread of the distances without plotting the outliers. It shows that the variation of the distances between levels is reduced after the projection. This is especially important when considering the stability of the ME reconstruction, since nonuniform sample sequences with varying large distances lead to numerical instability when solving (6). However, the projection to the $\tau$-domain slightly increases the overall median distance.

### C. Signal Reconstruction

To confirm the suggestion of the previous section that time-warping improves our sampling grid, we want to compare the reconstruction performance using ME reconstruction in the $t$-domain and the $\tau$-domain. Furthermore, we want to compare these results with the reconstruction performance using only the time-warped LC samples closest to the samples $y(t_n)$.

To reconstruct from our $\tau$-domain samples, we need to modify (5) [12]:

$$y(t_k) = \sum_{n=1}^{N} \boldsymbol{c}[n] \operatorname{sinc}(\tau_k - n) \tag{20}$$

and (7):

$$\boldsymbol{G}[k, n] = \operatorname{sinc}(\tau_k - n). \tag{21}$$

We can then plug the resulting $\boldsymbol{c}$ into the time-warping interpolation formula (9):

$$y(t) = \sum_{n=-\infty}^{\infty} c_n \cdot \operatorname{sinc}(\gamma(t) - n). \tag{22}$$

Fig. 5 shows the normalized mean squared error (NMSE) over the number of levels $N_L$. As the number of levels increases, the number of LC samples to reconstruct from also increases, thus reducing the reconstruction error in all three variants of the ME reconstruction. Furthermore, projection onto the $\tau$-domain drastically reduces the reconstruction error. If we now select only the LC samples closest to the sample times $t_n$ that fully describe a VBW signal, we discard a lot of information about our signal. Although the reconstruction performance decreases compared to reconstruction with all samples, it still improves sufficiently as more levels are added. This shows that for signal recovery, we are most interested in the LC samples closest in time to the signal samples $y(t_n)$.

## VII. Conclusion

This paper presents the advantage of projecting LC samples onto the $\tau$-domain prior to ME reconstruction for VBW signals with known instantaneous bandwidths. We show that projection reduces large distances between sampling times at the cost of a slightly higher median sampling distance. Furthermore, we compared the performance of ME reconstruction with and without time-warping, and with selected samples only, highlighting the reconstruction benefits. However, since the $B(t)$ of a signal is not known in practice, these results remain a best-case analysis. Due to the unique property of the LC sampler to sample proportionally to the instantaneous bandwidth, we expect a satisfactory estimate of $B(t)$ to be possible. There is an ongoing effort to estimate $B(t)$ directly from the LC samples of a signal [9], [16]. Using the information inherent to the sampling process is, in our opinion, a promising way to successfully recover signals from LC samples.

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