FAST SUM RATE MAXIMIZATION FOR THE DOWNLINK OF MIMO-OFDM SYSTEMS

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ABSTRACT

A new method for calculating the sum rate capacity of Gaussian MIMO-OFDM broadcast channels is proposed. It is based on the hybrid scheme from [1], which was shown to have better convergence properties than all other algorithms known from the literature. Our modification results in a further significant speed-up without the need for a heuristic parameter.

1. INTRODUCTION

MIMO-OFDM is a promising candidate for future wireless communication systems as it allows for very high data rates. The achievable sum rate is a reasonable optimization criterion as long as fairness among users is not an issue. For the uplink, all mobile stations usually have individual power constraints. The sum capacity is achievable by successive decoding and the optimum transmit covariance matrices can be determined by iterative waterfilling [2]. In the downlink, dirty paper coding is required instead. A direct optimization of the covariance matrices is difficult as the objective function is not concave. In [3], a duality between broadcast and multiple-access channels with sum power constraint was established, which allows to solve the simpler uplink problem and transform the solution for the downlink. However, the variable power allocation among users makes the task become more involved as the straightforward modification of iterative waterfilling for this case is not guaranteed to converge. A very simple algorithm that alternatingly optimizes the transmit covariance matrices and the power distribution was introduced in [4]. Recently, a hybrid scheme has been proposed [1], which seems to outperform all other existing sum rate optimization algorithms. It uses a fast, but not necessarily convergent method in order to generate a good starting point for an optimal, but slow one. Here, we suggest to include a low complexity approximate line search in each iteration. It will be demonstrated that our new approach results in an improved rate of convergence.

The remainder of this paper is organized as follows. The system model and the optimization problem is defined in Section 2. The mentioned sum rate maximization algorithms are reviewed in Section 3, before our extension is described in Section 4. Section 5 contains simulation results, and concluding remarks can be found in Section 6.

2. SYSTEM MODEL AND PROBLEM STATEMENT

Consider a MIMO-OFDM system with L subcarriers and K active users, where the base station is equipped with M antennas and user k has N_k antennas. The corresponding channel on the l-th subcarrier is defined by the $M \times N_k$ equivalent baseband channel matrix $\mathbf{H}_k[l]$, which is assumed to be perfectly known at both ends of the transmission link. Because of the duality mentioned before we will directly concentrate on the *uplink*. With the $N_k \times 1$ transmit vectors $\mathbf{x}_k[l]$ and the $M \times 1$ Gaussian noise vector $\mathbf{n}[l]$, the receive signal per subcarrier can be written as

$$\mathbf{y}[l] = \sum_{k=1}^{K} \mathbf{H}_k[l]\,\mathbf{x}_k[l] + \mathbf{n}[l]\ , \quad 1 \leq l \leq L\ . \tag{1}$$

Without loss of generality we assume uncorrelated noise with unit variance. Ignoring the spectral loss due to the guard interval, the maximum sum rate of this multiple-access channel (in bit per channel use) is given by

$$C_{\text{sum}} = \max_{\mathbf{Q}_k[l]} \frac{1}{L} \sum_{l=1}^{L} \log_2 \left| \mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k[l]\,\mathbf{Q}_k[l]\,\mathbf{H}_k^H[l] \right| \tag{2}$$

and equals the desired downlink sum capacity if the maximization is performed over all positive semi-definite transmit covariance matrices $\mathbf{Q}_k[l] = \mathrm{E}\left\{\mathbf{x}_k[l]\,\mathbf{x}_k^H[l]\right\}$ that fulfill the *total* power constraint

$$\frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{K} \mathrm{tr}\left\{\mathbf{Q}_k[l]\right\} \leq P\ . \tag{3}$$

The objective function in (2) is concave, so standard convex optimization algorithms [5] could be applied. More sophisticated approaches exploiting the special structure of the problem at hand are described in the next section.

Using successive decoding in the uplink and dirty paper coding in the downlink, the resulting noise covariance matrices (including interference) for user k are given by

$$\mathbf{M}_k = \mathbf{I} + \sum_{i=k+1}^{K} \mathbf{H}_i\,\mathbf{Q}_i\,\mathbf{H}_i^H\ , \quad \mathbf{B}_k = \mathbf{I} + \sum_{i=1}^{k-1} \mathbf{H}_k^H\,\mathbf{S}_i \mathbf{H}_k\ , \tag{4}$$

respectively, where the subcarrier index l was omitted for notational brevity. Then, introducing the economy size singular value decomposition of

$$\mathbf{M}_k^{-\frac{1}{2}}\,\mathbf{H}_k\,\mathbf{B}_k^{-\frac{1}{2}} = \mathbf{U}_k\,\mathbf{\Sigma}_k\,\mathbf{V}_k^H \tag{5}$$

with the *square* diagonal matrix $\mathbf{\Sigma}_k$ containing only the nonzero singular values, the solution to (2) can be transformed in order to obtain the optimum $M \times M$ downlink covariance matrices [3]

$$\mathbf{S}_k = \mathbf{M}_k^{-\frac{1}{2}} \mathbf{U}_k \mathbf{V}_k^H \mathbf{B}_k^{\frac{1}{2}} \mathbf{Q}_k \mathbf{B}_k^{\frac{1}{2}} \mathbf{V}_k \mathbf{U}_k^H \mathbf{M}_k^{-\frac{1}{2}} \tag{6}$$

that also fulfill the sum power constraint (3).

3. SUM RATE MAXIMIZATION ALGORITHMS

In this section we review some existing algorithms for the sum rate maximization in multiuser MIMO systems. As the required operations are identical for all subcarriers, we restrict to $L = 1$ for simplicity and omit the index l in order to make the presentation more readable.

3.1. Individual Power Constraints

Let us first consider the case where the users have individual power constraints

$$\mathrm{tr}\{\mathbf{Q}_k\} \leq P_k\ , \quad 1 \leq k \leq K\ . \tag{7}$$

This is typical for an uplink scenario. Assume that user k treats *all* other users as colored noise, hence the corresponding noise covariance matrix is given by

$$\mathbf{Z}_k = \mathbf{I} + \sum_{i \neq k} \mathbf{H}_i \mathbf{Q}_i \mathbf{H}_i^H \tag{8}$$

Then the objective function in (2) can be rewritten as

$$\begin{aligned} &\log_2 \left| \mathbf{Z}_k + \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H \right| \\ &= \log_2 \left| \mathbf{I} + \mathbf{Q}_k \mathbf{H}_k^H \mathbf{Z}_k^{-1} \mathbf{H}_k \right| + \log_2 |\mathbf{Z}_k|\ , \end{aligned} \tag{9}$$

where the determinant properties $|\mathbf{I} + \mathbf{AB}| = |\mathbf{I} + \mathbf{BA}|$ and $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ have been used. Defining the eigenvalue decomposition of the *effective* channel covariance matrix

$$\mathbf{H}_k^H \mathbf{Z}_k^{-1} \mathbf{H}_k = \mathbf{W}_k \mathbf{\Lambda}_k \mathbf{W}_k^H \tag{10}$$

it is easy to verify that the maximization of (9) over $\mathbf{Q}_k$ while all other transmit covariance matrices are fixed leads to the well-known waterfilling solution [6]

$$\mathbf{Q}_k = \mathbf{W}_k \mathbf{P}_k \mathbf{W}_k^H \tag{11}$$

$$\text{with } \mathbf{P}_k = \max\left(\mu_k \mathbf{I} - \mathbf{\Lambda}_k^{-1}\ ,\ \mathbf{0}\right)\ , \tag{12}$$

where max denotes an element-wise maximum here, and the waterfilling level μ_k is chosen such that the power constraint (7) is fulfilled. Modifying $\mathbf{Q}_k$ affects the noise seen by the other users. Hence, the transmit covariance matrices must be updated successively for $k = 1, \ldots, K$ using (8), (10) and (11) in an iterative fashion [2]. As single-user waterfilling can never decrease the sum rate, this procedure is guaranteed to converge to the global optimum. The pseudo-code is shown in Fig. 1.

Initialization
$\mathbf{Q}_k = \frac{P_k}{N_k}\mathbf{I}$
$\mathbf{R} = \mathbf{I} + \sum_{k=1}^K \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H$

repeat
  for $k = 1, \ldots, K$
    Eigenvalue decomposition of effective channel
    $\mathbf{Z}_k = \mathbf{R} - \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H$
    $\mathbf{H}_k^H \mathbf{Z}_k^{-1} \mathbf{H}_k = \mathbf{W}_k \mathbf{\Lambda}_k \mathbf{W}_k^H$

    Single-user waterfilling
    $\mathbf{P}_k = \max\left(\mu_k \mathbf{I} - \mathbf{\Lambda}_k^{-1}, \mathbf{0}\right) \quad \text{s.t. } \mathrm{tr}\{\mathbf{P}_k\} = P_k$

    Update transmit and receive covariance matrix
    $\mathbf{Q}_k = \mathbf{W}_k \mathbf{P}_k \mathbf{W}_k^H$
    $\mathbf{R} = \mathbf{Z}_k + \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H$
  end
until convergence

Fig. 1. Iterative waterfilling with individual power constraints

3.2. Alternating Power and Covariance Optimization

In [4], the algorithm in Fig. 2 was proposed for the sum rate maximization with total power constraint. Given an initial power distribution P_k with $\sum_{k=1}^K P_k = P$, the covariance matrices $\mathbf{Q}_k$ or, equivalently, their normalized versions $\bar{\mathbf{Q}}_k = \mathbf{Q}_k / P_k$ are optimized using the iterative waterfilling from the previous section. Usually, a single iteration suffices in this step. Then, keeping $\bar{\mathbf{Q}}_k$ fixed, the power distribution is optimized. For this purpose, standard interior point methods can be used because the objective function in (2) is concave with respect to P_k. These two steps are alternatingly repeated until convergence. Although this approach always converges to the global optimum unless a user is switched off prematurely, it does not fully exploit the structure of the optimization problem. This is especially critical for large number of users.

Initialization
$P_k = P/K$

repeat
  Optimize normalized covariance matrices $\bar{\mathbf{Q}}_k$
  for fixed power distribution P_k
  using iterative waterfilling with individual constraints

  Optimize power distribution P_k
  for fixed normalized covariance matrices $\bar{\mathbf{Q}}_k$
  using standard interior point methods
until convergence

Fig. 2. Alternating power and covariance optimization

3.3. Sum Power Iterative Waterfilling

Looking at the Karush-Kuhn-Tucker optimality conditions [5], the only difference between individual and total power constraints is that in the latter case the waterfilling level must be the same for all users, i.e. $\mu_k = \mu$. Thus, instead of updating the covariance matrices $\mathbf{Q}_k$ one after another, the waterfilling procedure should be performed for all users simultaneously as demonstrated in Fig. 3. However, in Section 3.1 the effective channel of the currently optimized user does not change by the power reallocation. This is not true anymore if all transmit covariance matrices are modified at the same time. As a consequence, the sum rate does not necessarily increase after simultaneous waterfilling. In fact, convergence could only be proven for $K = 2$ users in [1].

Initialization
$\mathbf{Q}_k = \frac{P}{K \cdot N_k} \mathbf{I}$
$\mathbf{R} = \mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k \, \mathbf{Q}_k \, \mathbf{H}_k^H$

repeat
Eigenvalue decomposition of all effective channels
$\mathbf{Z}_k = \mathbf{R} - \mathbf{H}_k \, \mathbf{Q}_k \, \mathbf{H}_k^H$
$\mathbf{H}_k^H \, \mathbf{Z}_k^{-1} \, \mathbf{H}_k = \mathbf{W}_k \, \mathbf{\Lambda}_k \, \mathbf{W}_k^H$

Simultaneous waterfilling for all users
$\mathbf{P}_k = \max\left(\mu \mathbf{I} - \mathbf{\Lambda}_k^{-1}, \mathbf{0}\right) \quad \text{s.t.} \ \sum_{k=1}^{K} \operatorname{tr}\{\mathbf{P}_k\} = P$

Update transmit and receive covariance matrices
$\mathbf{Q}_k = \mathbf{W}_k \, \mathbf{P}_k \, \mathbf{W}_k^H$
$\mathbf{R} = \mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k \, \mathbf{Q}_k \, \mathbf{H}_k^H$
until convergence

Fig. 3. Iterative waterfilling with total power constraint

3.4. Cyclic Coordinate Ascent

Waterfilling with individual power constraints is a special case of the globally convergent cyclic coordinate ascent algorithm [1]. In order to apply it with total power constraint let us consider the optimization problem

$$\begin{aligned} &\max_{\mathbf{Q}_k^{(j)}} \frac{1}{K} \sum_{\kappa=1}^{K} \log_2 \left| \mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k \, \mathbf{Q}_k^{([k-\kappa]_K+1)} \, \mathbf{H}_k^H \right| \\ &\text{s.t.} \ \sum_{k=1}^{K} \operatorname{tr}\left\{\mathbf{Q}_k^{(j)}\right\} = P \,, \quad 1 \le j \le K \,. \end{aligned} \tag{13}$$

From the concavity of the objective function it follows that the optimum covariance matrices must fulfill $\mathbf{Q}_k^{(j)} = \mathbf{Q}_k$, $1 \le j \le K$, so (13) is obviously equivalent to the original sum rate maximization (2). Observe that each index j appears only once per summand. Hence, all $\mathbf{Q}_k^{(j)}$ can be iteratively updated for fixed j using sum power waterfilling without changing the current effective channels. A major drawback is the required memory, as K covariance matrices need to be stored per user. This can be avoided if additionally after each waterfilling step the covariance matrices

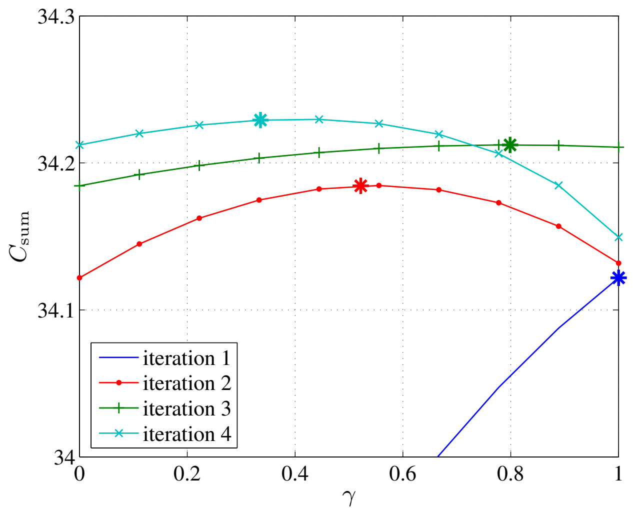

Fig. 4. Example for the relation between sum rate and γ

are averaged. Then in the sum power iterative waterfilling algorithm shown in Fig. 3, only the transmit covariance update formula needs to be replaced by

$$\mathbf{Q}_k = \frac{1}{K} \cdot \mathbf{W}_k \, \mathbf{P}_k \, \mathbf{W}_k^H \; + \; \frac{K-1}{K} \cdot \mathbf{Q}_k \,. \tag{14}$$

For large number of users convergence is very slow due to the small factor in front of the first, updated term. Therefore, a hybrid scheme was also suggested in [1], where the fast algorithm based on (11) is only used for a finite number of iterations before switching to the convergent version (14).

4. EXTENSION BY EFFICIENT LINE SEARCH

The hybrid algorithm appears to be somewhat heuristic. Note, however, that (11) and (14) can be generalized to

$$\mathbf{Q}_k = \gamma \cdot \mathbf{V}_k \, \mathbf{P}_k \, \mathbf{V}_k^H \; + \; (1-\gamma) \cdot \mathbf{Q}_k \tag{15}$$

with $1/K \le \gamma \le 1$. Instead of abruptly changing γ from its maximum to the minimum value, we suggest to perform a simple line search in order to maximize the sum rate in each iteration. To this end, standard bisection methods could be employed. However, the example shown in Fig. 4 for the system parameters given in Section 5 demonstrates that the functional relation between the sum rate and the parameter γ can be well approximated by a parabola, i.e.

$$C_{\mathrm{sum}}(\gamma) \approx a\gamma^2 + b\gamma + c \,. \tag{16}$$

Solving for a and b and setting the first derivative to zero we obtain

$$\gamma_{\mathrm{opt}} \approx \min\left(\frac{3\,C_{\mathrm{sum}}(0) - 4\,C_{\mathrm{sum}}(\frac{1}{2}) + C_{\mathrm{sum}}(1)}{4\,C_{\mathrm{sum}}(0) - 8\,C_{\mathrm{sum}}(\frac{1}{2}) + 4\,C_{\mathrm{sum}}(1)}, 1\right), \tag{17}$$

which is indicated by the stars in Fig. 4. It matches quite well with the true optimum. Note that $C_{\mathrm{sum}}(0)$ equals $C_{\mathrm{sum}}(\gamma_{\mathrm{opt}})$ from the previous iteration and does not need to be recalculated. Furthermore, $C_{\mathrm{sum}}(1)$ is the sum rate achieved by the original sum power iterative waterfilling; it actually decreases in the fourth iteration while $C_{\mathrm{sum}}(1/K)$ is considerably smaller than $C_{\mathrm{sum}}(\gamma_{\mathrm{opt}})$.

Initialization

$\mathbf{Q}_k = \frac{P}{K \cdot N_k} \mathbf{I}$

$\mathbf{R} = \mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k \, \mathbf{Q}_k \, \mathbf{H}_k^H$

repeat

Eigenvalue decomposition of all effective channels

$\mathbf{Z}_k = \mathbf{R} - \mathbf{H}_k \, \mathbf{Q}_k \, \mathbf{H}_k^H$

$\mathbf{H}_k^H \, \mathbf{Z}_k^{-1} \, \mathbf{H}_k = \mathbf{W}_k \, \mathbf{\Lambda}_k \, \mathbf{W}_k^H$

Simultaneous waterfilling for all users

$\mathbf{P}_k = \max\left(\mu \mathbf{I} - \mathbf{\Lambda}_k^{-1}, \mathbf{0}\right) \quad \text{s.t.} \ \sum_{k=1}^{K} \operatorname{tr}\{\mathbf{P}_k\} = P$

Line search for optimum weighting factor

$\mathbf{Q}_k^{\text{new}} = \mathbf{W}_k \, \mathbf{P}_k \, \mathbf{W}_k^H$

$\mathbf{R}^{\text{new}} = \mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k \, \mathbf{Q}_k^{\text{new}} \, \mathbf{H}_k^H$

$C_{\text{sum}}(\gamma) = \log_2 |\gamma \cdot \mathbf{R}^{\text{new}} + (1-\gamma) \cdot \mathbf{R}|$

Determine γ_{opt} from (17)

Update transmit and receive covariance matrices

$\mathbf{Q}_k = \gamma_{\text{opt}} \cdot \mathbf{Q}_k^{\text{new}} + (1-\gamma_{\text{opt}}) \cdot \mathbf{Q}_k$

$\mathbf{R} = \gamma_{\text{opt}} \cdot \mathbf{R}^{\text{new}} + (1-\gamma_{\text{opt}}) \cdot \mathbf{R}$

until convergence

Fig. 5. Iterative waterfilling with efficient line search

5. SIMULATION RESULTS

In order to demonstrate the convergence behavior of the different sum rate maximization algorithms, a set of channel matrices $\mathbf{H}_k[l]$ was randomly generated from a complex Gaussian distribution. The number of subcarriers was fixed to $L = 8$, and the base station with $M = 4$ antennas has a total power constraint of $P = 20$ dB. There were $K = 15$ users having $N_k = 2^n$ antennas for $5n+1 \leq k \leq 5(n+1)$, respectively. The results are depicted in Fig. 6.

As individual power constraints $P_k = P/K$ are more restrictive, the sum rate is smaller in this case. For the alternating algorithm, only one Newton step is performed during power optimization in order to keep the complexity per iteration more or less comparable for all methods. This explains the extremely slow convergence. The sum power iterative waterfilling from Section 3.3 shows good performance at the beginning, but then fails to converge. On the other hand, the modified updating rule (14) derived from the cyclic coordinate ascent algorithm with an additional averaging step allows only for very small changes of the transmit covariance matrices in each iteration due to the large number of users K. For the hybrid scheme we switched to the convergent update formula after 5 iterations as suggested in [1]. At this point, the original version has already started diverging. Off course, this can be avoided by adaptive switching. However, unless the sum rate actually decreases, it may be difficult to decide when to switch. This problem does not arise if a simple line search is included. It always outperforms the hybrid scheme as it (at least approximately) maximizes the sum rate in each iteration requiring only a negligible computational overhead. Our new algorithm converges in less than 10 iterations to the global optimum.

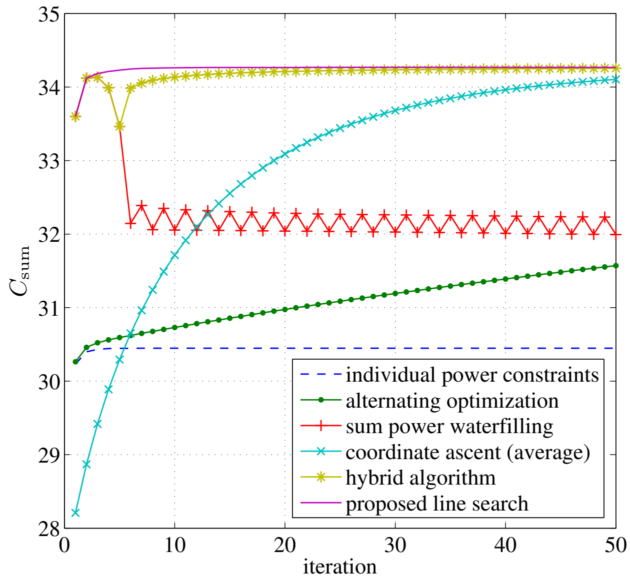

Fig. 6. Convergence behavior of sum rate maximization algorithms

6. CONCLUSION

In this paper, several algorithms for the maximization of the sum rate in the downlink of MIMO-OFDM systems were reviewed. They all make use of a duality in order to solve an easier uplink problem with total power constraint. The alternating approach of [4] does not fully exploit the structure of the objective function. A hybrid scheme proposed in [1] performs much better. We demonstrated that including a simple line search in each iteration can further increase the speed of convergence significantly.

7. REFERENCES

[1] N. Jindal, W. Rhee, S. Vishwanath, S. Jafar, and A. Goldsmith, "Sum power iterative waterfilling for multi-antenna Gaussian broadcast channels," *IEEE Transactions on Information Theory*, 2005, to appear.

[2] W. Yu, W. Rhee, S. Boyd, and J. Cioffi, "Iterative waterfilling for Gaussian vector multiple-access channels," *IEEE Transactions on Information Theory*, vol. 50, no. 1, pp. 145–152, January 2004.

[3] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates, and sum-rate capacity of MIMO broadcast channels," *IEEE Transactions on Information Theory*, vol. 49, no. 10, pp. 2658–2668, October 2003.

[4] E. Jorswieck and H. Boche, "Sum capacity optimization of the MIMO gaussian MAC," in *Proc. WPMC*, Honolulu, USA, 2002.

[5] S. Boyd and L. Vandenberghe, "Convex optimization," Stanford University," course reader, 2002.

[6] I. Telatar, "Capacity of multi-antenna gaussian channels," *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585–595, November 1999.