Applicability of Importance Sampling for Rayleigh Fading Mobile Radio Channels

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Abstract: Estimating the bit error rate (BER) of complex digital communication systems has often been realised by using the Monte Carlo method. For very low BER, this method requires a high computational effort. Therefore, a lot of developments in theory and application of importance sampling (IS) techniques have been done especially for AWGN channels (additive white gaussian noise) or time-varying uniformly distributed channels.

In this paper we discuss the applicability of several IS techniques for the simulation of Rayleigh fading mobile radio channels. By using the system threshold characteristic (STC) we first make an analytical statement of the IS efficiency for an AWGN and a frequency-nonselective Rayleigh fading channel. It is shown that due to the Rayleigh distribution the efficiency is reduced. Furthermore, the so called overbiasing effect is graphically explained and simulation results illustrate that the theoretically possible efficiencies cannot be reached in practice because of this statistic effect.

1. INTRODUCTION

The recent and future demand for mobile communication systems has increased the importance of suitable system models and simulation techniques for estimating the performance of these systems. A common definition of performance is the bit error rate (BER) and the traditional way of extracting a numerical estimate of the BER by simulation is the Monte Carlo method. For low BER this method requires excessive run time and therefore, importance sampling (IS) techniques are proposed to reduce this time.

In the literature several IS techniques have been introduced [1]-[4] assuming that uniformly distributed signals are corrupted by white gaussian noise. Extremely large speed up factors have been recognised.

The aim of this paper is to answer the question if IS is applicable for the simulation of Rayleigh fading channels. Therefore, IS is investigated for Rayleigh distributed instead of uniformly distributed signals. In particular, we compare theoretically the efficiency of IS for an AWGN-channel and a frequency-nonselective Rayleigh fading channel by evaluating the STC. Besides these analytical derivations, the implementation effect of overbiasing the parametric scheme of the system is graphically explained. Subsequently, simulation results make clear that already for a frequency-nonselective Rayleigh fading channel the theoretically possible efficiency of IS cannot be reached because of the overbiasing effect. Additionally, the influence of the time-selectivity has been analysed by simulating frequency-nonselective Rayleigh fading channels with different fading rapidities.

After introducing basic IS formulations, the STC and the overbiasing effect are described. Finally, simulation results illustrate the applicability of IS for Rayleigh fading channels.

2. SYSTEM MODEL

Fig. 1 shows the block diagram of a discrete equivalent baseband system model used in this paper.

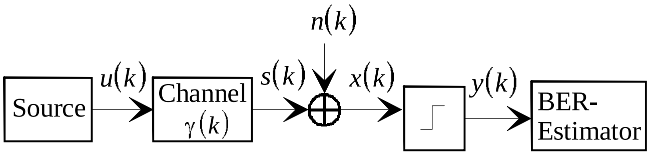

Fig. 1: Block diagram of a equivalent baseband system

A binary information source generates every T seconds a symbol $u(k)$ (k indicates the symbol interval) with equally likely amplitudes $\pm A$. These symbols are fed to a discrete memoryless channel. In the case of an AWGN channel, $s(k)$ can be expressed as $s(k) = u(k)$ whereas for a frequency-nonselective Rayleigh fading channel $s(k) = \gamma(k) \cdot u(k)$ holds. The channel coefficient

$$\gamma(k) = \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-j\omega_{d,n} kT} \right| \tag{1}$$

is modeled by summing up N pointers rotating in the complex plane with N different Doppler frequencies $\omega_{d,n}$, where N is the number of transmission paths. The Doppler frequencies are identically Jake distributed in the interval $[-2\pi f_{d\max}, 2\pi f_{d\max}]$. For a large number N of transmission paths the random process $\gamma(k)$ is Rayleigh distributed

$$f_\Gamma(\gamma) = \begin{cases} \frac{\gamma}{\sigma^2} \cdot \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) & \text{for } \gamma \geq 0 \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

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It is important to note that due to (1) the values $\gamma(k_1)$ and $\gamma(k_2)$ are correlated for different time instances k_1 and k_2. The signal $s(k)$ is then corrupted by additive white gaussian noise $n(k)$ to obtain the input signal $x(k)=s(k)+n(k)$ of the thresholder. Finally, the quantized values $y(k)=\pm A$ are fed into the BER estimator.

3. IS-FORMULATION

The error probability of the assumed communication system can be written as

$$P_b = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{H}[s+n] f_S(s) f_N(n) ds\, dn, \tag{3}$$

where $f_N(n)$ and $f_S(s)$ are the probability density functions (pdf) of $n(k)$ and $s(k)$, respectively. $\mathrm{H}[\cdot]$ is an indicator function with its output equal to 1 when the detected symbol is erroneous, and 0 otherwise. It is assumed that all random processes of the system are ergodic and that $s(k)$ and $n(k)$ are statistically independent for every instant k. In order to apply IS to the noise source $n(k)$, P_b is given by

$$P_b = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{H}[s+n] f_s(s) f_N^*(n) w(n) ds\, dn \tag{4}$$

where

$$w(n) = \frac{f_N(n)}{f_N^*(n)}. \tag{5}$$

According to the considered IS technique, different modified probability density functions $f_N(n)$ can be used. Assuming an unbiased pdf of the noise

$$f_N(n) \in N\left(\mu = 0, \sigma^2\right), \tag{6}$$

CIS (conventional importance sampling) is defined as

$$f_N^*(n) \in N\left(\mu = 0, \sigma_*^2\right) \quad \text{with} \quad \delta = \frac{\sigma_*^2}{\sigma^2} > 1 \tag{7}$$

and IIS (improved importance sampling) as

$$f_N^*(n) \in N\left(\mu = c, \sigma^2\right) \quad \text{with} \quad c \neq 0, \tag{8}$$

where $N(\cdot)$ is the gaussian pdf with mean μ and variance σ^2. The variance reduction factor of CIS is denoted by δ and the amount of translating the pdf for IIS by c.

The empirical IS-estimator related to (4) can be formulated as

$$\hat{P}_b = \frac{1}{K}\sum_{k=1}^{K} \mathrm{H}[s_k + n_k] w(n_k) \tag{9}$$

where s_k and n_k are values of the random processes $s(k)$ and $n(k)$ at any time instant k. K is any large integer number depending on the desired accuracy of the estimation.

For systems with a frequency-nonselective Rayleigh fading channel it is also possible to modify the Rayleigh distribution by decreasing the Rayleigh variance instead of biasing the noise source. This method is called CIS-Rayleigh and the corresponding variance factor is denoted by $\beta = \sigma_{R*}^2 / \sigma_R^2$ where σ_R^2 and σ_{R*}^2 are the unbiased and biased variances of the Rayleigh process, respectively.

4. APPLICABILITY

4.1. System Threshold Characteristic (STC)

As known, the basic idea of importance sampling is to reduce the variance of the BER estimation by artificially increasing the error probability. Obviously, the random processes $s(k)$ and $n(k)$ in Fig. 1 are the relevant components affecting the BER. In order to obtain a statement of the IS efficiency for different channels, it is necessary to find a characteristic criterion of the system including all important components. The so called System Threshold Characteristic (STC)

$$T(n) = \int_{-\infty}^{\infty} \mathrm{H}[s+n] f_S(s) ds \tag{10}$$

has been used in [5] to find the optimal biasing scheme and is also suitable to approach this problem. Equation (10) represents the conditional mean of $\mathrm{H}[s+n]$ over the random process $s(k)$ given a noise value n. Since $\mathrm{H}[s+n]$ is the error indicator function, the STC determines the probability of error for the distribution of the signal process $s(k)$ under the condition of a given noise value n.

Since the STC represents a probability it follows that $0 \le T(n) \le 1$. The region corresponding to this interval is called transition region [5]. In the following discussion a theoretical comparison of the efficiency of IS for an AWGN and a frequency-nonselective Rayleigh fading channel is performed by evaluating the STC of these two channels.

AWGN channel

In the case of an AWGN channel $s(k)=u(k)$ holds so that the pdf of the signal process $s(k)$ can be written by using the delta distribution $\delta(\cdot)$ as

$$f_S(s) = \frac{1}{2}\cdot[\delta(s-A)+\delta(s+A)]. \tag{11}$$

By substituting (11) in (10) and being aware of the different indicator functions for different transmitted symbols, the STC can be expressed as

$$T(n) = \begin{cases} \int_{-\infty}^{\infty} \mathrm{H}_A[s+n]\delta(s-A)ds & u = A, \forall n \\ \int_{-\infty}^{\infty} \mathrm{H}_{-A}[s+n]\delta(s+A)ds & u = -A, \forall n \end{cases} \tag{12}$$

$$= \begin{cases} \varepsilon(-A-n) & u = A \\ \varepsilon(n-A) & u = -A \end{cases}$$

where $\varepsilon(\cdot)$ represents the unit step function.

The evaluated STC is illustrated in Fig. 2 as a function of the noise values n. Additionally, the unbiased density function $f_N(n)$ of the noise process is shown to explain the properties of IS.

Concerning the efficiency of IS, it is significant to note, that the transition regions only include the points

n=+A and n=-A. In other words, the transition regions are arranged in a distance A from the origin. Assuming a high signal to noise ratio, involving a high amplitude A and a narrow pdf $f_N(n)$, the great majority of noise values possesses zero error probability $(T(n)=0)$. Therefore, it is possible to increase the BER extremely by translating the pdf $f_N(n)$ or raising its variance.

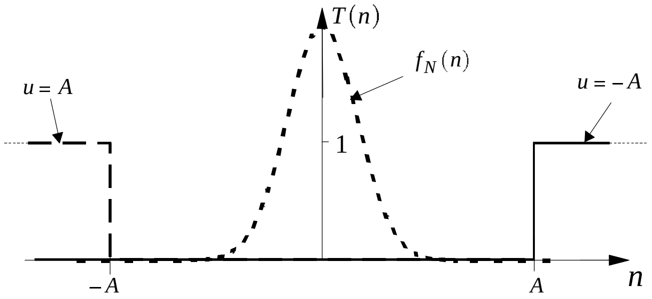

Fig. 2: STC and pdf $f_N(n)$ for an AWGN channel

Furthermore, the characteristic shape of the STC ($dT/dn \to \infty$ for n=+/-A) provides a high speed up factor even for small values of δ and c. Therefore, the arrangement of the smallest transition regions being possible in a distance A from the origin ensures the tremendous runtime improvement of several decades of IS for an AWGN channel. This result is often illustrated in the literature, for example in [1] - [4].

Frequency-nonselective Rayleigh fading channel

Here, $s(k)$ is determined by the product of the statistically independent random processes $u(k)$ and $\gamma(k)$. The aim to find the pdf of $s(k)$ can be reached by using the relationship

$$f_S(s) = \int_{-\infty}^{\infty} \frac{1}{|u|} f_U(u) f_\Gamma\left(\frac{s}{u}\right) du, \tag{13}$$

which finally yields to the expression

$$f_S(s) = \frac{1}{A} \begin{cases} f_\Gamma\left(\frac{s}{A}\right) & u = A \\ f_\Gamma\left(\frac{s}{-A}\right) & u = -A \end{cases}. \tag{14}$$

By substituting (14) into (10) the STC of the frequency-nonselective Rayleigh fading channel can be expressed in the form

$$T(n) = \begin{cases} F_\Gamma\left(\frac{-n}{A}\right) & u = A, n < 0 \\ F_\Gamma\left(\frac{n}{A}\right) & u = -A, n > 0, \\ 0 & \text{else} \end{cases} \tag{15}$$

where $F_\Gamma(x) = \int_{-\infty}^{x} f_\Gamma(y) dy = 1 - \exp\left(-x^2/2\sigma^2\right)$ represents the cumulative Rayleigh distribution function. Fig. 3 shows the STC for the transmitted symbols +/-A and the unbiased pdf of the noise process.

First of all, one can recognise that the transition regions are directly arranged at the origin. As mentioned before, a distance between the origin and the transition regions ensures a high efficiency. Hence, it is possible to make the qualitative statement that the runtime improvement of IS for a frequency-nonselective Rayleigh fading channel is not as high as the one for an AWGN channel.

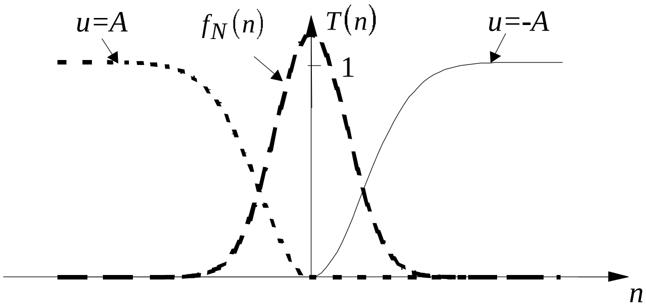

Fig. 3: STC and $f_N(n)$ for a frequency-nonselective Rayleigh fading channel

Furthermore, the gradient dT/dn of the STC is determined by the variance of the Rayleigh distribution in (15). This property indicates that biasing the Rayleigh process by modifying the corresponding variance σ_R^2 might be more efficient than biasing the noise process. This IS method is named CIS-Rayleigh. As can be seen in fig. 3, increasing the BER requires a higher gradient dT/dn and therefore it is necessary to decrease the Rayleigh variance. Thus, CIS-Rayleigh is rendering both, the growth of the efficiency and the efficiency itself.

Finally, it is significant to note, that all results mentioned are only valid for an infinite number of samples K for the IS estimation of the BER. For a finite number of samples, the so called Overbiasing effect appears and it will be seen that especially for a high S/N the theoretically possible efficiency cannot be reached.

4.2. Overbiasing Effect

This section presents the very important effect of overbiasing the parametric scheme of a system by implementing IS. Using the IIS technique, it has been mathematically proven in [6] that for a finite number of samples and large amounts of biasing an underestimation of the BER occurs. Particularly, for a given number of samples K the BER estimates can be made arbitrarily small by increasing the amount of biasing (see (16)).

$$P\left\{\lim_{c\to\infty} \hat{P}_b < W\right\} \to 1 \quad \text{for any value } W > 0. \tag{16}$$

This relationship stated in (16) between the BER estimation and the biasing procedure is called Overbiasing Effect and the corresponding region of c Overbiasing Region. For explaining this effect we focus our attention on the IS estimator $\hat{P}_b$ and therefore on the IS weights $w(n_k)$. Assuming the IIS technique the weights can be expressed in the form

$$w(n) = \frac{f_N(n)}{f_N^*(n)} = \exp\left[-\frac{\left(2nc - c^2\right)}{2\sigma^2}\right]. \tag{17}$$

Fig. 4 shows the weights $w(n)$ and the biased density function $f_N^*(n)$ for different amounts of translation c and an assumed variance $\sigma^2 = 0.5$. For an increasing magnitude of translation c the values n_k with extremely low weights $w(n_k)$ occur more frequently. Assuming an IIS simulation with a finite number of samples, it could happen that only values n_k with very small weights will be considered. Here, the mean of the estimation $\hat{P}_b$ decreases by increasing the amount of translation. At this point, it is visible that the finite number of samples is responsible for the underestimation of the BER, whereas for theoretical considerations the statement of an unbiased IS estimation with the property $\mathrm{E}\{\hat{P}_b\} = P_b$ still holds.

If CIS is considered, the explanation is similar and confirmed by the simulation results illustrated in the following chapter. It should only be mentioned that because of a smaller transition of the weights, the effect just appears for very large reduction factors δ.

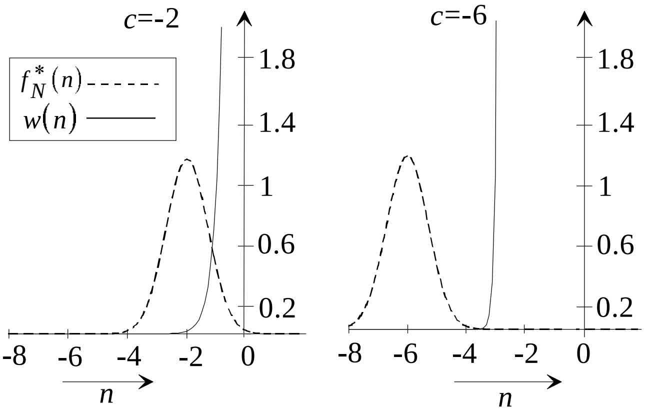

Fig. 4: $w_{\mathrm{IIS}}(n)$ and $f_N^*(n)$ for $c=-2$, $c=-6$

5. SIMULATION RESULTS

5.1. Simulation parameters

In order to apply IS for a time-varying Rayleigh fading channel with mutually correlated output values s_k the simulation process for each BER estimation was divided into a number of short subsimulations. For each subsimulation new, randomly chosen Doppler frequencies were used to calculate the channel coefficient $\gamma(k)$. As a result, the statistic of the channel was well represented and the values s_k for different subsimulations were mutually uncorrelated. If not explicitly mentioned, the data rate was chosen to be R=270 kbit/s and the maximum Doppler frequency was selected to f_{dmax}=100 Hz ($2 f_{dmax} T$=7.4·10^{-4}<<1 → time-nonselective channel). The system performance and the variance of the estimation were calculated according to the unbiased estimators

$$\overline{\hat{P}_b} = \frac{1}{J} \sum_{j=1}^{J} \hat{P}_{b,j} \tag{18}$$

and

$$S^2 = \frac{1}{J-1} \sum_{j=1}^{J} \left(\hat{P}_{b,j} - \overline{\hat{P}_b} \right)^2 \tag{19}$$

for J subsimulations, respectively. $\hat{P}_{b,j}$ is the estimated BER for the j-th subsimulation given by (8) with K samples. For the frequency-nonselective channel, the analytical BER is known so that K was computed based on a 95% confidence interval $[1/3 P_b \; 5/3 P_b]$. For all simulations, the efficiency of IS was calculated by $\eta = S_{MC}^2 / S_{IS}^2$, whereby the index MC indicates the Monte Carlo simulation.

5.2. Rayleigh distribution and Overbiasing Effect

Fig. 5 illustrates the obtained efficiencies as a function of c using IIS with different values E_b/N_0. Additionally, Fig. 6 shows the corresponding bit error rates.

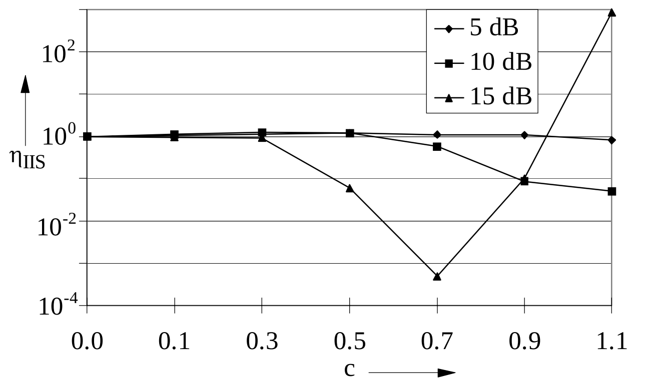

Fig. 5: η_{IIS} as a function of c for different E_b/N_0

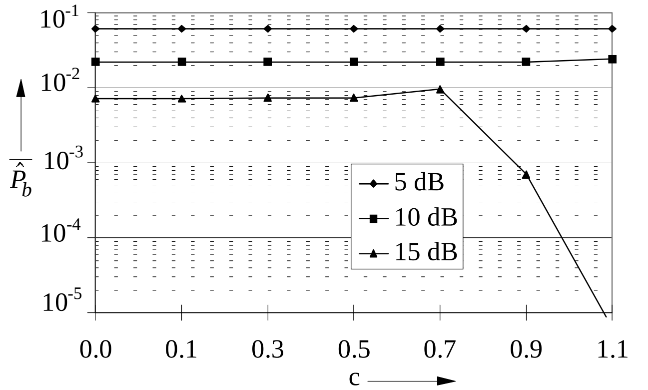

Fig. 6: $\overline{\hat{P}_b}$ as a function of c for different E_b/N_0

First of all, it can be seen that for a signal to noise ratio of 5 dB no runtime improvement of IIS over MC can be achieved. For higher signal to noise ratios like 15 dB the efficiency is decreasing for small amounts of translation and it finally growths for larger amounts. This typical behavior of the IIS estimator is based on the overbiasing effect which results in the deterioration of the BER estimation as shown in Fig. 6. This deterioration occurs especially for high signal to noise ratios like 15 dB and therefore in cases where IIS should gain the highest runtime improvements.

Fig. 7 shows the obtained efficiencies $\eta_{\mathrm{CIS-R}}$ for CIS-Rayleigh. It can be recognised that this method leads in general to a performance gain depending on the underlying signal to noise ratio. For example, at $E_b/N_0 \,\hat{=}\, 15\,\mathrm{dB}$ one can reach a runtime improvement of merely a factor 10 by reducing the variance by $\beta \approx 5\cdot$

10^{-2}. However, overbiasing the parametric scheme also yields a deterioration of the estimation.

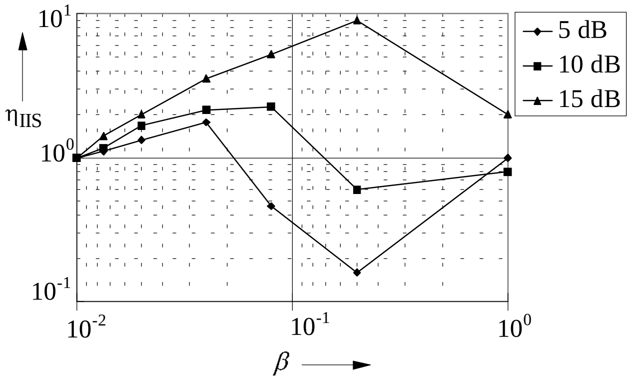

Fig. 7: η_{CIS-R} as a function of the variance factor β

5.3. Influence of time-selectivity

Since the time-selectivity of the channel plays a significant role concerning the applicability of IS we complete our investigations by analyzing the IS efficiencies for different fading rates. Tab. 1 shows the efficiencies for a time-nonselective Rayleigh fading channel and a channel with independent samples drawn from a Rayleigh distributed density at a very high signal to noise ratio of 43 dB.

$E_b/N_0 \hat{=} 43\,\mathrm{dB}$	η_{CIS-R}	β
time-nonselective channel $2f_{dmax}T$=$7.4 \cdot 10^{-4}$	24	$5 \cdot 10^{-3}$
time-selective channel $2f_{dmax}T \to \infty$	$2 \cdot 10^3$	$2.5 \cdot 10^{-4}$

Tab. 1: Comparison of time-selective and time-nonselective Rayleigh fading channel

These results illustrate the tremendous effect of time-selectivity for IS-implementations. To get a deeper comprehension, Table 2 illustrates the simulation results obtained for different maximum Doppler frequencies f_{dmax} for a specific variance factor $\beta = 10^{-3}$ and $E_b/N_0 \hat{=} 43\,\mathrm{dB}$.

f /Hz	$2f_{dmax}T$	$\eta_{CIS\text{-}R}$
50	$3.7 \cdot 10^{-4}$	16
100	$7.4 \cdot 10^{-4}$	24
200	$1.5 \cdot 10^{-3}$	30
500	$3.7 \cdot 10^{-3}$	80

Tab. 2: IS-efficiencies for a time-nonselective fading channel ($E_b/N_0 \hat{=} 43\,\mathrm{dB}$)

Obviously, the maximum Doppler frequency affects directly the runtime improvement. Besides the constraint of representing the statistic of the channel one should be aware that the time-nonselective fading implicates correlated samples during each subsimulation. It is shown in [3] that correlated samples increase the variance of the IS estimator and therefore decrease the efficiency.

6. CONCLUSION

This article deals with the applicability of IS methods for the simulation of Rayleigh fading mobile radio channels. It was shown that for a Rayleigh distributed process the efficiency is not as high as for a uniformly distributed process implicated by an AWGN-channel. The best performance was obtained by modifying the variance of the Rayleigh process whereas biasing the noise yielded no runtime improvements. Furthermore, for illustrating the difference between theoretical and simulation results the important effect of overbiasing the parametric scheme was graphically explained for a gaussian distributed noise process. Finally, the detrimental influence of correlated consecutive samples was illustrated by comparing a slowly fading channel and a channel with statistical independent output samples. Simulation results confirme that the theoretically possible efficiency cannot be reached due to the overbiasing effect.

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