SUPPRESSION OF GAUSSIAN NOISE USING CUMULANTS: A QUANTITATIVE ANALYSIS

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ABSTRACT

Higher-Order-Statistics (HOS) are being used in many areas of digital signal processing, e.g. in the field of array processing. The main aim is often to suppress Gaussian noise. Mostly, the corresponding algorithms are applied to short data blocks, because only then the stationarity of the data needed for cumulant estimation is given. In many cases, not enough attention is paid to the fact that for short data blocks the suppression of Gaussian noise is small compared to the estimation error made because of the higher order of the cumulants. In this paper, the property of cumulants to suppress Gaussian noise is studied in detail. With an algorithm for direction-of-arrival (DOA) estimation in the field of array processing, the estimation errors that occur when using HOS are compared with the estimation errors that occur when using 2nd order statistics. A quantitative result will be given to show that for short data blocks the suppression of Gaussian noise with HOS doesn't lead to a better result than using 2nd order statistics.

1. INTRODUCTION

The use of Higher-Order-Statistics (HOS) for estimation problems is getting more important in the recent years. With HOS it is possible to derive more information from the received signal than with 2nd order statistics, which have commonly been used before. For example, the phase of the transmission channel can not be derived from the symbol-rate sampled (stationary) received signal if conventional second order statistics are used only. This information can only be derived with HOS [1] [2] or by exploiting cyclostationarity of the data [3] [4]. Cyclostationarity will not be dealt with in this paper.

Another benefit of HOS, which is often the main reason of using it, is the suppression of Gaussian noise, for example, in the field of array processing [5] [6]. But in many cases not enough attention is paid to the fact that for short data blocks the suppression of Gaussian noise is small compared to the estimation error made because of the higher order of cumulants. Just for increasing data length, the suppression of Gaussian noise of HOS becomes more important. In realistic situations, however, we can only work with short data blocks (about several hundred samples) in order to meet the stationarity assumption made for the estimation.

To clarify the benefit of HOS for the suppression of Gaussian noise quantitative experiments are carried out in this paper. In section 2, estimation errors of cumulants of white Gaussian noise are studied. In section 3, a distorted signal with Gaussian noise is used to analyse the influence of the suppression of Gaussian noise on the estimated cumulants of the distorted signal. In section 4, an algorithm for DOA estimation in array processing is used to show the disadvantage of HOS for short data blocks if no compensation is made for the estimation errors of the cumulants used. In section 5 the conclusion of this paper follows.

Other advantages of HOS in array processing, e.g. the extension of the effective array [7] [8], the calibration of the steering vector in a one signal case [9], and the improvement of the estimation variance of the cumulants through averaging over all possible ways to compute a correlation [8], will not be dealt with in this paper.

2. SUPPRESSION OF GAUSSIAN NOISE USING CUMULANTS

For the analysis of the suppression of Gaussian noise by cumulants, stationary white complex Gaussian noise $r(n)$ is used. Theoretically, with total suppression of Gaussian noise $r(n)$, the autocumulants of $r(n)$ should be zero.

$$\mathrm{cum}_4^r(\lambda_1, \lambda_2, \lambda_3) = 0 \tag{1}$$

Unfortunately, we can get this ideal result only if the data length used for the estimation is infinite. This condition is not met in realistic cases.

To show the dependence of the estimated autocumulants $\widehat{\mathrm{cum}}_4^r(\lambda_1, \lambda_2, \lambda_3)$ on the data length, the following test has been executed: 12 different data lengths between 50 and

Matlab programs and compressed postscript files of our publications are readily available from our WWW server `http://www.comm.uni-bremen.de`.

1000 samples are used to estimate the autocumulants by unbiased sample averaging in 500 Monte-Carlo-Runs, where the parameter λ_1 is set to values ranging from -19 to 20 and the parameters λ_2 and λ_3 are set to zero. Since the exact values of the autocumulants are zero, the estimation results of $\widehat{\text{cum}}_4^r(\lambda_1, \lambda_2, \lambda_3)$ are equal the estimation errors.

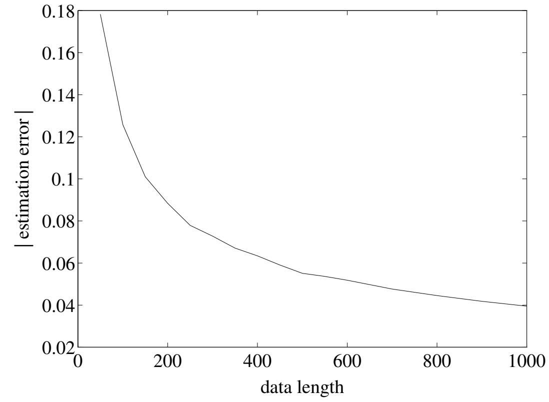

Figure 1: Cumulant estimation errors of a Gaussian process in terms of data length

In Figure 1, the estimation errors of $\widehat{\text{cum}}_4^r(\lambda_1, \lambda_2, \lambda_3)$ are averaged over all parameters λ_1, λ_2 and λ_3. The dependence of the estimation errors on the data lengths is shown. With growing data length the estimation error is getting smaller.

In order to show the influence of the suppression of Gaussian noise on the estimation of cumulants of a distorted signal, a further test is executed in section 3.

3. ANALYSIS OF THE INFLUENCE OF THE SUPPRESSION OF GAUSSIAN NOISE ON THE ESTIMATION OF CUMULANTS

For the analysis of the influence of the suppression of Gaussian noise on the estimaton of cumulants, the model given in Figure 2 is used in this paper:

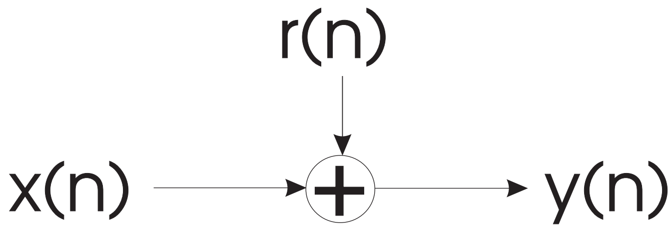

Figure 2: System model for the analysis of cumulant estimation errors

Here, $x(n)$ is a zero mean, non-Gaussian, i.i.d. QPSK signal and $r(n)$ is stationary additive white Gaussian noise. $x(n)$ and $r(n)$ are statistically independent. $y(n)$ is the sum of both processes.

For the analysis, we estimate the autocumulants of $y(n)$. Because of the finite data length used for the estimation of the autocumulants, $\widehat{\text{cum}}_4^y(\lambda_1, \lambda_2, \lambda_3)$ can not be equal to the exact values $\text{cum}_4^y(\lambda_1, \lambda_2, \lambda_3)$. There is always an estimation error $\Delta_4^y(\lambda_1, \lambda_2, \lambda_3)$.

$$\begin{aligned} \widehat{\text{cum}}_4^y(\lambda_1, \lambda_2, \lambda_3) \quad &= \quad \text{cum}_4^y(\lambda_1, \lambda_2, \lambda_3) + \\ & \quad\quad \Delta_4^y(\lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{2}$$

As shown in Figure 2, $y(n)$ is the sum of the signal $x(n)$ and the Gaussian noise $r(n)$, where $x(n)$ and $r(n)$ are statistically independent. So we can express the total estimation error $\Delta_4^y(\lambda_1, \lambda_2, \lambda_3)$ as:

$$\Delta_4^y(\lambda_1, \lambda_2, \lambda_3) = \Delta_4^x(\lambda_1, \lambda_2, \lambda_3) + \Delta_4^r(\lambda_1, \lambda_2, \lambda_3) \tag{3}$$

Here, $\Delta_4^x(\lambda_1, \lambda_2, \lambda_3)$ is the estimation error caused by the finite data length of the undistorted signal $x(n)$ and $\Delta_4^r(\lambda_1, \lambda_2, \lambda_3)$ is the estimation error caused by Gaussian noise.

To show the influence of the suppression of Gaussian noise on the estimation of cumulants, the ratio of $\Delta_4^y(\lambda_1, \lambda_2, \lambda_3)$ and the true autocumulant $\text{cum}_4^y(\lambda_1, \lambda_2, \lambda_3)$, which is equal to $\text{cum}_4^x(\lambda_1, \lambda_2, \lambda_3)$, is used.

$$f(\lambda_1, \lambda_2, \lambda_3) = \frac{|\Delta_4^y(\lambda_1, \lambda_2, \lambda_3)|}{|\text{cum}_4^x(\lambda_1, \lambda_2, \lambda_3)|} \tag{4}$$

Here data lengths ranging from 100 to 8000 are used to compute f. The same values of λ and number of Monte-Carlo-Runs as in section 2 are used for Figure 3. Since $f(\lambda_1, \lambda_2, \lambda_3)$ is dependent on the signal-to-noise ratio (SNR), different SNR's have been used for the test.

As a comparison, the relative estimation errors of the correlation estimates of $y(n)$, which are similarly defined as in equation 2, are also shown in Figure 3.

As far as the relative estimation errors are concerned, we realize from Figure 3 that there is a certain minimum blocklength (depending on the noise power) below which cumulant estimation errors exceed the correlation estimation errors.

In realistic applications we mostly work with short data blocks (about 200 samples) to meet the stationarity assumption of a data block for the estimation. Moreover, the SNR values that we can assume are usually between 5dB and 10dB in mobile radio applications, e.g. . So it is interesting to know whether the use of HOS rather than 2nd order statistics for an estimation problem leads to a better result on these conditions because of the advantage of HOS of suppressing Gaussian noise. Often, no general statement can be made to answer the question how critical a given relative estimation error proves to be for the respective algorithm.

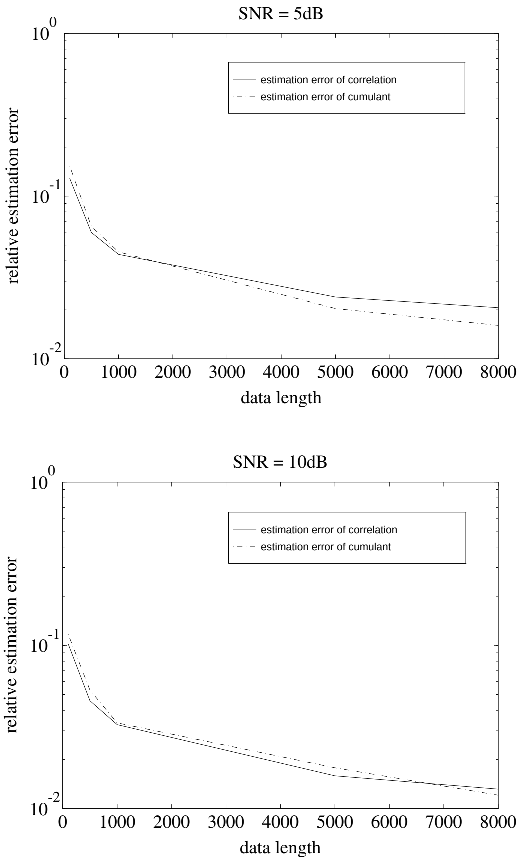

Figure 3: Influence of the suppression of Gaussian noise on the error of cumulant estimates of a distorted signal: (a) SNR = 5dB; (b) SNR = 10dB

In the following section the cumulant and correlation estimation errors will be compared with an algorithm for direction-of-arrival (DOA) estimation in array processing.

4. COMPARISON OF THE CUMULANT AND CORRELATION ESTIMATION ERRORS WITH A DOA ESTIMATION ALGORITHM

The ESPRIT algorithm is a common approach for the DOA (direction-of-arrival) estimation of incoming signals in array processing, which is based on the eigenvalue decomposition of the spatial correlation matrix of the sensors' output signals. [10]. A modified version of this algorithm was presented in [8]. It uses 4th order statistics and is called VESPA.

For the simulations, two independent QPSK sources (i.e. zero mean, non-Gaussian and i.i.d. processes) are used to illuminate a linear array consisting of 8 equispaced omnidirectional sensors. The DOA's of these sources are $\phi_1 = 20^0$ and $\phi_2 = 40^0$ with respect to the array normal. The additive sensor noise is assumed to be white and Gaussian which is independent between the sensors and independent from the sources.

In order to compare the estimation errors of VESPA and ESPRIT, only 5 sensors are used for VESPA, which correspond to 8 sensors in ESPRIT by using the HOS advantage of the extention of the effective array. The mean square errors of the estimated DOA's with these two algorithms are used for the comparison.

$$\mathrm{MSE}_4 = \frac{1}{500}\sum_{i=1}^{500}\sum_{j=1}^{2}|\widehat{\phi}_j^{VESPA}(i) - \phi_j(i)|^2 \tag{5}$$

$$\mathrm{MSE}_2 = \frac{1}{500}\sum_{i=1}^{500}\sum_{j=1}^{2}|\widehat{\phi}_j^{ESPRIT}(i) - \phi_j(i)|^2 \tag{6}$$

Here, $\widehat{\phi}_j^{VESPA}(i)$ is the estimated DOA of the j-th source in the i-th Monte-Carlo-Run with VESPA and $\widehat{\phi}_j^{ESPRIT}(i)$ is the estimated DOA of the j-th source in the i-th Monte-Carlo-Run with ESPRIT.

Since the estimation errors of both algorithms are dependent on SNR, two cases are compared for SNR = 5dB and SNR = 10dB.

In Figure 4 it is clear to see that a data length of at least 5000 samples is needed to get a better estimation result with HOS than with 2nd order statistics. So we can say that the suppression of Gaussian noise by HOS leads to better DOA estimates only from a certain large data length onwards which is unrealistic in many cases. Therefore, we can not confirm the results presented in [8] for blocklengths of 100 samples.

5. CONCLUSION

This paper has given a quantitative analysis of the Gaussian noise suppression property of HOS. By a DOA estimation example, it has been shown that for short data lengths, the use of HOS only because of their advantage to suppress Gaussian noise is not to be recommended.

6. REFERENCES

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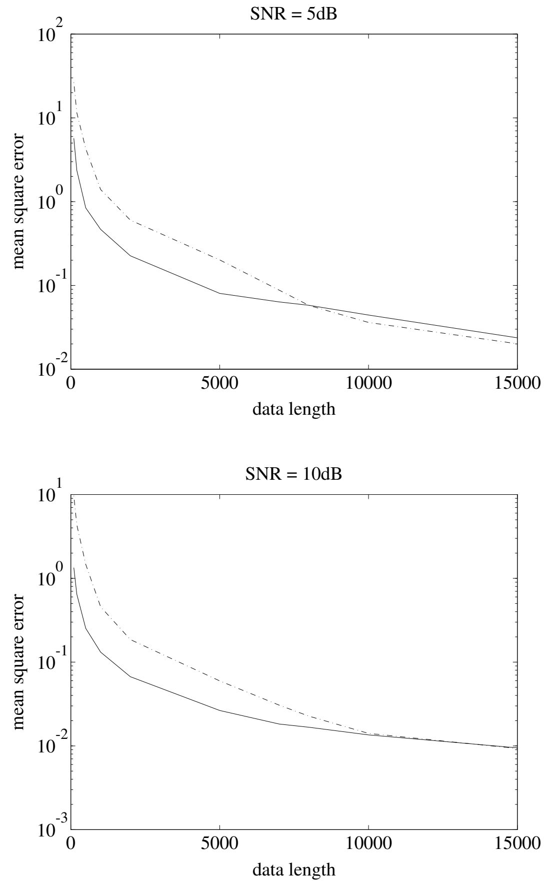

Figure 4: Comparison of the estimation errors using ESPRIT (solid lines) and VESPA (dash-dotted) at: (a) SNR = 5dB; (b) SNR = 10dB

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