Channel Coding 2

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Lecture

Tuesday, 08:30 – 10:00 in N2420

Exercise

Wednesday, 14:00 – 16:00 in N2420

Dates for exercises will be announced during lectures.

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Outline Channel Coding II

- 1. Concatenated Codes
  - Serial Concatenation
  - Parallel Concatenation (Turbo Codes)
  - Iterative Decoding with Soft-In/Soft-Out decoding algorithms
  - EXIT-Charts
- 2. Trelliscoded Modulation (TCM)
  - Motivation by information theory
  - TCM of Ungerböck, pragmatic approach by Viterbi, Multilevel codes
  - Distance properties and error rate performance
  - Applications (data transmission via modems)
- 3. Adaptive Error Control
  - Automatic Repeat Request (ARQ)
  - Performance for perfect and disturbed feedback channel
  - Hybrid FEC/ARQ schemes

Channel Coding I:

- Different schemes for error detection and error correction
  - Adding redundancy so that only part of all possible sequences are transmitted
  - Distance between valid sequences is increased
  - Due to added redundancy error detection or correction is possible

Motivation for Coded Modulation:

- The bandwidth per user of several channels is restricted (e.g. 3 kHz telephone)
- Uncoded BPSK provides small data rate – additional coding would further reduce the transmission rate
  → impossible to achieve reliable communication with high data rates

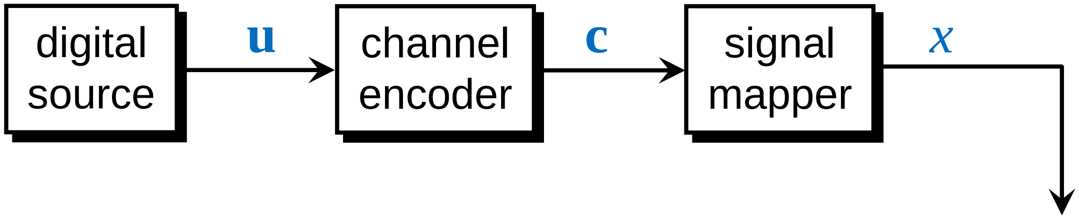
One Possible Solution:

- **Coded Modulation**: Combining **channel coding** and **higher order modulation schemes**

Chapter 2. Trelliscoded Modulation

- Linear Digital Modulation
  - Basics and Minimum Euclidean distance
  - Spectral efficiency and error rate performance of linear modulation schemes
- Principle of Coded Modulation
  - Basic approach
  - Capacity of AWGN Channel for Different Linear Digital Modulation Schemes
- TCM by Ungerböck
  - First approaches towards Trelliscoded Modulation
  - Set-Partitioning
  - Principal structure of TCM encoders and optimal codes of Ungerböck
- ML-Decoding with Viterbi-Algorithm
- Analytical Approximation of Bit Error Probability
- Pragmatic Approach by Viterbi
- Multilevel Codes by Imai
- TCM for telephone modems

Structure of Digital Transmission System



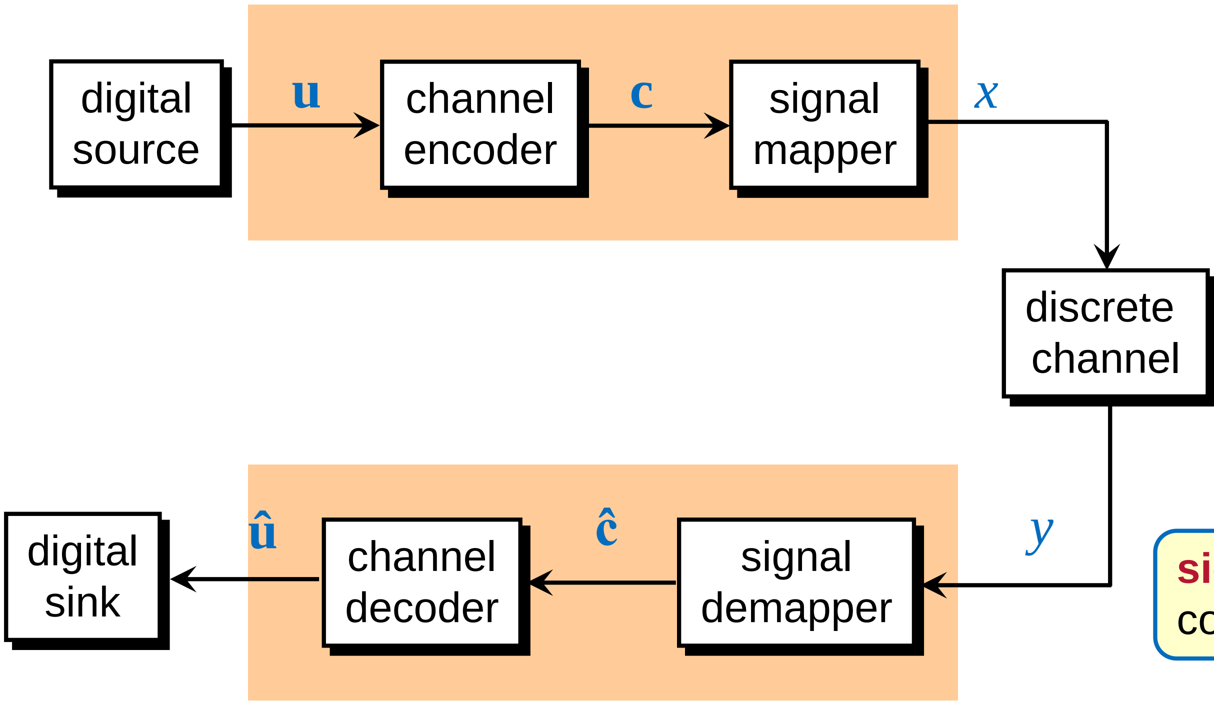
Digital Source comprises analog source and source coding, delivers digital data vector $\mathbf{u} = [u_1\ u_2\ \dots\ u_k]$ of length k at symbol clock T_s

Channel encoder adds redundancy to $\mathbf{u}$ so that errors in $\mathbf{c} = [c_0\ c_1\ \dots\ c_{m\text{-}1}]$ can be detected or even corrected

- Channel encoder may consist of several constituent codes
- Code rate: $R_c = k\ /\ m$

signal mapper assigns m-bit vector $\mathbf{c}$ onto one out of $M = 2^m$ symbols x

Structure of Digital Transmission System



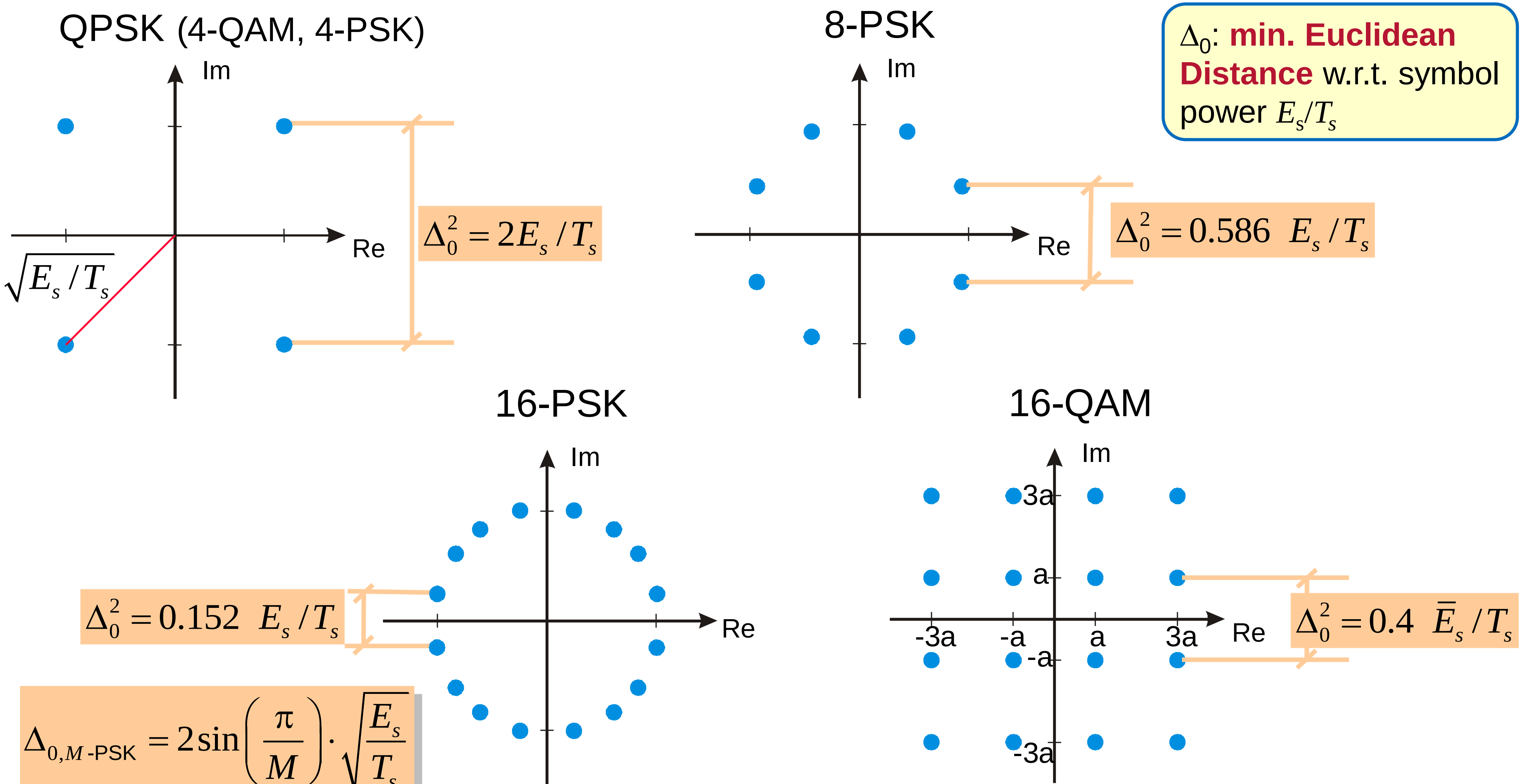
Discrete channel: comprises
- analog part of modulator
- physical channel
- analog part of demodulator

signal demapper represents counterpart for signal mapper

Channel decoder:
- Estimation of $\mathbf{u}$ given the received vector $\hat{\mathbf{c}}$
- $\hat{\mathbf{c}}$ doesn't necessarily consist of hard quantized values {0,1}
- Since encoder may consist of several parts, decoder may also consist of several modules

Linear Digital Modulation: Minimum Euclidean Distance

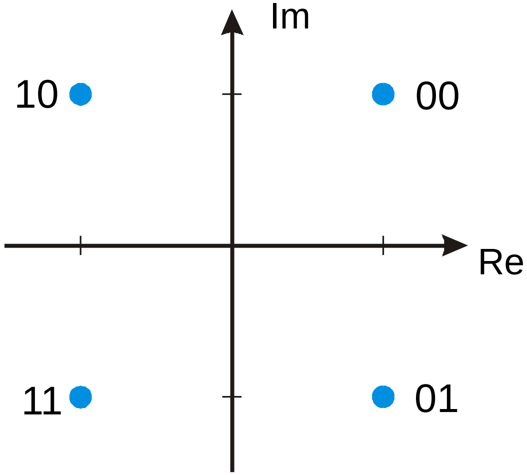
Δ_0: **min. Euclidean Distance** w.r.t. symbol power E_s/T_s



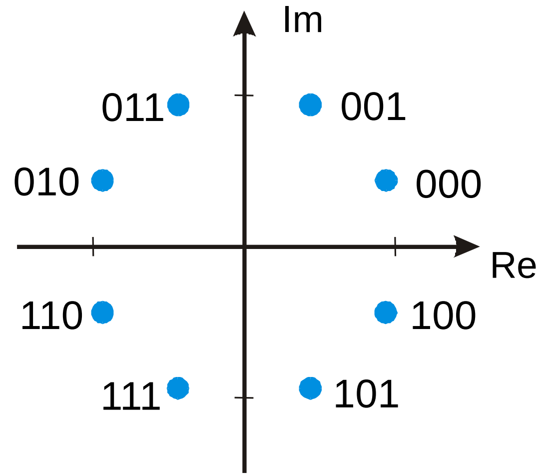
$$\Delta_{0,M\text{-PSK}} = 2\sin\left(\frac{\pi}{M}\right)\cdot\sqrt{\frac{E_s}{T_s}}$$

Linear Digital Modulation: Gray Mapping

4-QAM, 4-PSK (QPSK)

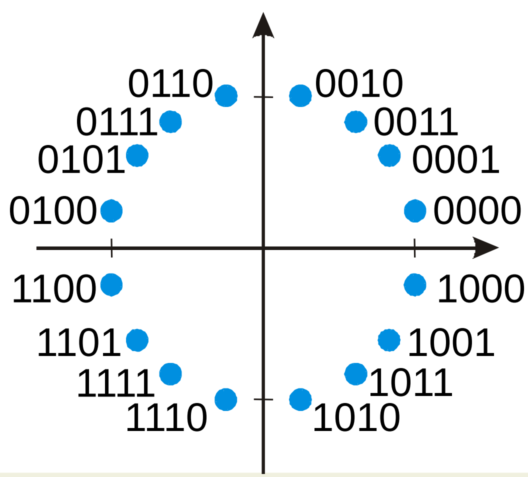


8-PSK

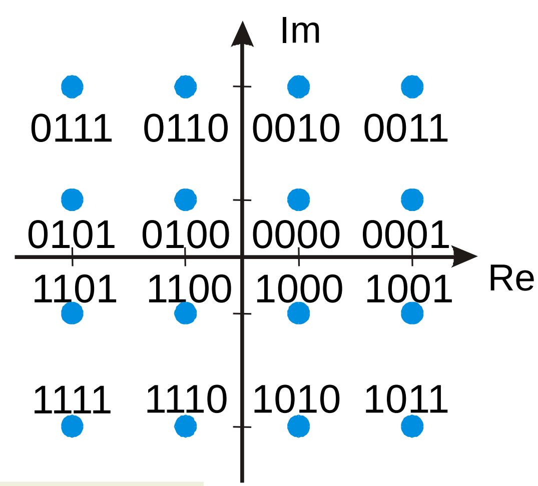


Gray Mapping: binary representations of neighboring symbols differ in only one bit
→ achieves minimum BER for uncoded transmission

16-PSK



16-QAM



Linear Digital Modulation: Impulse Shaping

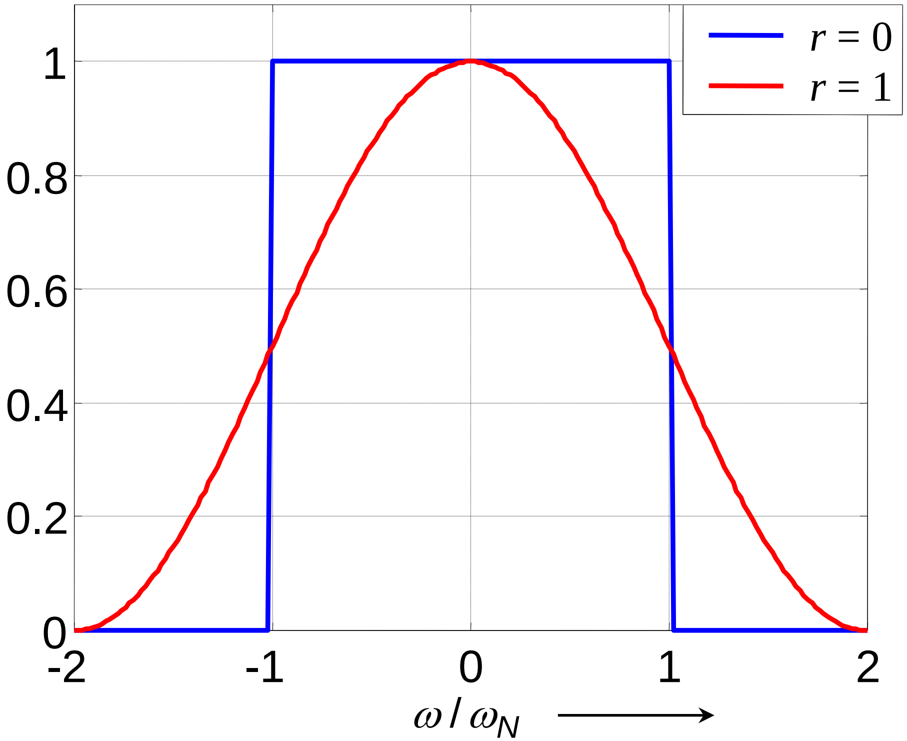
- **Bandwidth efficiency** depends on modulation size and **impulse filter** $g_r(t)$
- **Symbol rate** r_s leads to symbol duration $T_s = 1/r_s$
- **Bandwidth** $B = \alpha / T_s$

impulse filter $g_r(t)$ determines parameter α

- **Raised Cosine Filter:** parameter r
  - Impulse response $$g_r(t) = \frac{\sin(\omega_N t)}{\omega_N t} \cdot \frac{\cos(r\omega_N t)}{1-(4rf_N t)^2} = \mathrm{si}(\omega_N t) \cdot \frac{\cos(r\omega_N t)}{1-(4rf_N t)^2}$$
  - Transfer function $$G_r(j\omega) = \begin{cases} 1 & \frac{|\omega|}{\omega_N} \le 1-r \\ 0 & \frac{|\omega|}{\omega_N} \ge 1+r \\ \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2r}\left(\frac{|\omega|}{\omega_N} + r - 1\right)\right) & \text{else} \end{cases}$$

Nyquist frequency $f_N = 1/(2T_s)$

Linear Digital Modulation: Impulse Shaping



- Ideal low-pass filter ($r = 0$)

$$G_{r=0}(j\omega) = \begin{cases} 1 & \frac{|\omega|}{\omega_N} \le 1 \\ 0 & \text{else} \end{cases}$$

$$B = 2f_N = 1/T_s = r_s \quad \rightarrow \alpha = 1$$

- Cosine face ($r = 1$)

$$G_{r=1}(j\omega) = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}\frac{\omega}{\omega_N}\right) & \frac{|\omega|}{\omega_N} \le 2 \\ 0 & \text{else} \end{cases}$$

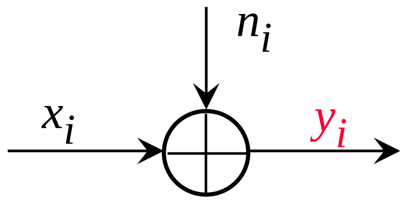
$$B = 4f_N = 2/T_s = 2r_s \quad \rightarrow \alpha = 2$$

spectral efficiency: $\eta = \frac{\text{info bit rate}}{\text{bandwidth}} = \frac{r_b}{B} = \frac{1/T_b}{\alpha / T_s}$ bit/symbol bit/s/Hz

In the sequel we assume always ideal low pass filter with $\alpha = 1$!

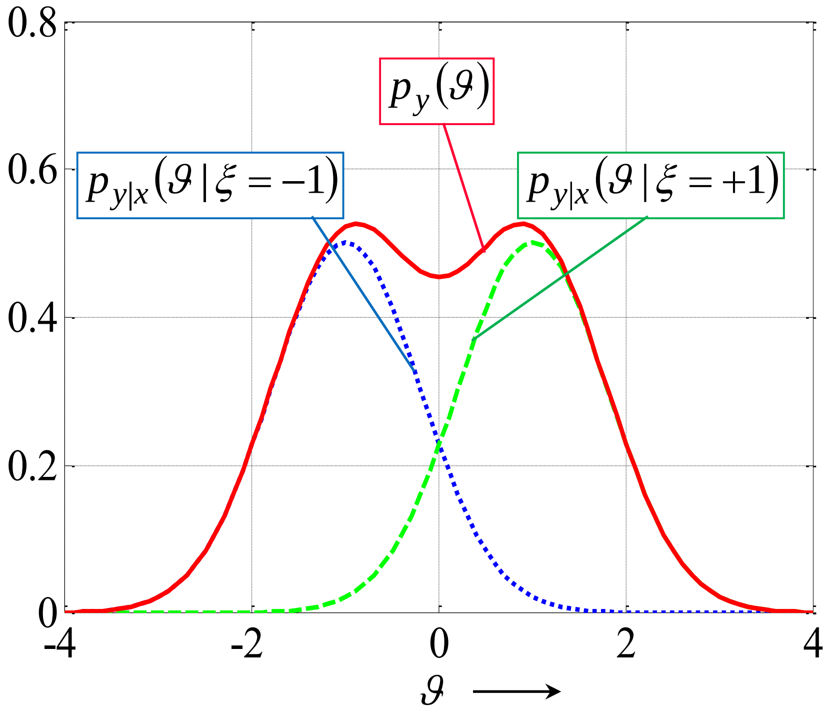
$$\eta_{r=0} = \frac{T_s}{T_b} = m$$

Linear Digital Modulation: BPSK and AWGN Channel

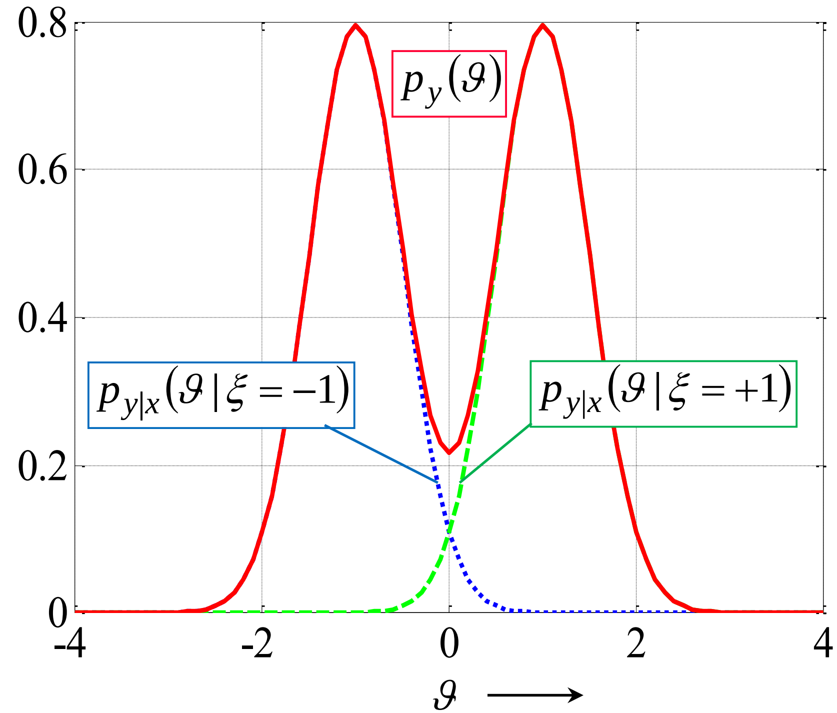


$$p_{y|X_\upsilon}(y) = \frac{1}{\sqrt{2\pi\sigma_N^2}} \cdot \exp\left[-\frac{(y - X_\upsilon)^2}{2\sigma_N^2}\right]$$

signal-to-noise-ratio $E_s/N_0 = 2$ dB



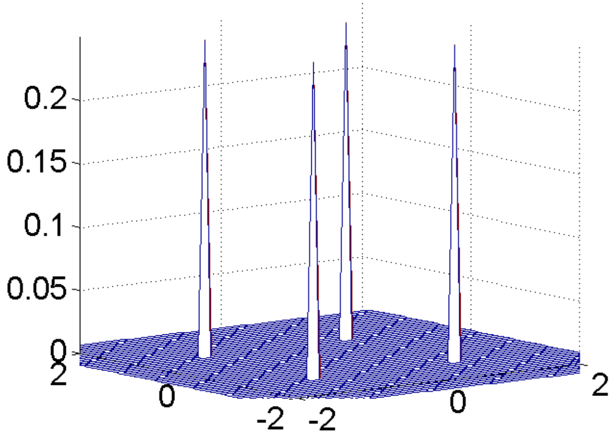
signal-to-noise-ratio $E_s/N_0 = 6$ dB



Linear Digital Modulation: QPSK and AWGN Channel

channel input (QPSK)

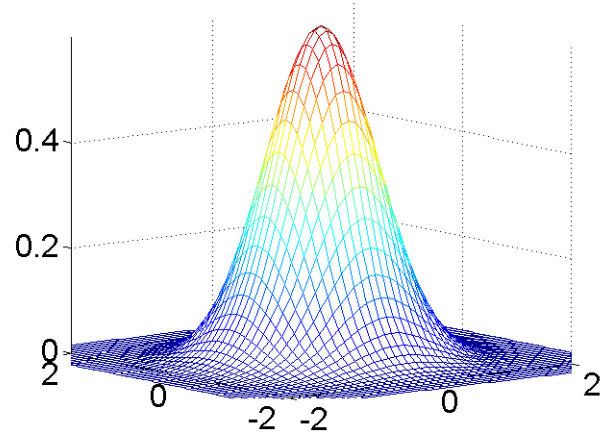
$$p_x(\xi) = \frac{1}{4} \cdot \sum_{\nu=0}^{3} \delta(\xi - X_\nu)$$



AWGN channel

$$p_n(\eta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\eta_I^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\eta_Q^2}{2\sigma^2}}$$

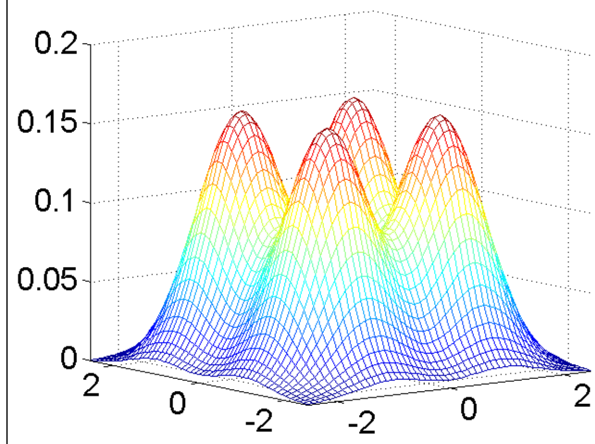
$$= \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{(\eta_I^2 + \eta_Q^2)}{2\sigma^2}}$$



channel output

$$p_y(\eta) = p_x(\xi) * p_n(\xi)$$

$$= \frac{1}{4} \cdot \sum_{\nu=0}^{3} p_n(\eta - X_\nu)$$



Linear Digital Modulation: Error Rate Performance

- Maximum likelihood criterion for symbol detection

$$\hat{x}_i = \arg\max_{X_\nu} p_{Y|X_\nu}\left(y_i \middle| x_i = X_\nu\right) = \arg\min_{X_\nu} \left|y_i - X_\nu\right|^2$$

- Symbol error probability

$$P_s = \Pr\left\{\left|y_i - x_i\right|^2 > \left|y_i - x_i'\right|^2\right\} \quad \forall\, x_i, x_i' \in \mathrm{A}_{in}, x_i \neq x_i'$$

- Error probability is dominated by minimum Euclidean distance Δ_0
  - *M*-PSK: $\Delta_0 = 2 \cdot \sin\left(\pi / M\right) \cdot \sqrt{E_s / T_s}$
  - *M*-ASK: $\Delta_0 = \sqrt{\frac{12}{M^2 - 1} \cdot \frac{\bar{E}_s}{T_s}}$ *M*-QAM: $\Delta_0 = \sqrt{\frac{6}{M - 1} \cdot \frac{\bar{E}_s}{T_s}}$

Minimum distance decreases and error rate increases with growing *M* !

Linear Digital Modulation: Error Rate Performance

- Error probability for M-ASK

$$P_s^{M\text{-ASK}} \approx \frac{M-1}{M} \cdot \operatorname{erfc}\left(\sqrt{\frac{3}{M^2-1} \cdot \frac{E_s}{N_0}}\right) = \frac{M-1}{M} \cdot \operatorname{erfc}\left(\sqrt{\frac{3m}{M^2-1} \cdot \frac{E_b}{N_0}}\right)$$

- Error probability for M-QAM (equivalent to squared $\sqrt{M}$-ASK)

$$P_s^{M\text{-QAM}} \approx 1-\left(1-P_s^{\sqrt{M}\text{-ASK}}\right)^2 = 2 \cdot P_s^{\sqrt{M}\text{-ASK}} - \left(P_s^{\sqrt{M}\text{-ASK}}\right)^2$$

$$< 2\frac{\sqrt{M}-1}{\sqrt{M}} \cdot \operatorname{erfc}\left(\sqrt{\frac{3m}{2(M-1)} \cdot \frac{E_b}{N_0}}\right) \quad \text{for} \quad E_s^{\sqrt{M}\text{-ASK}} = E_s^{M\text{-QAM}}$$

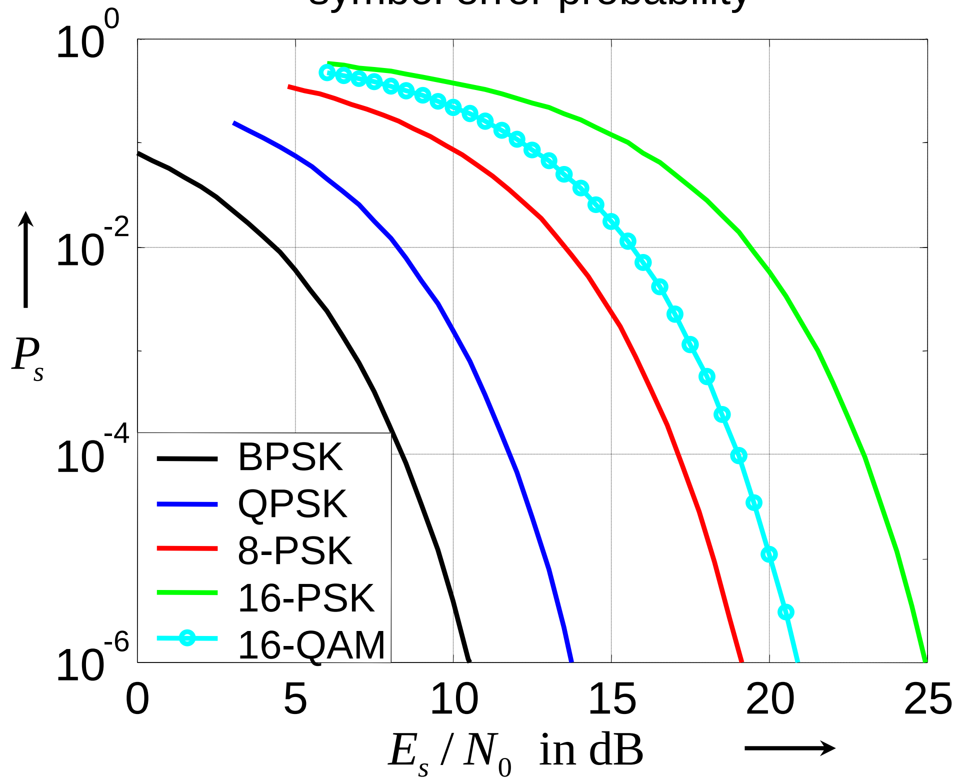
- Error probability for M-PSK

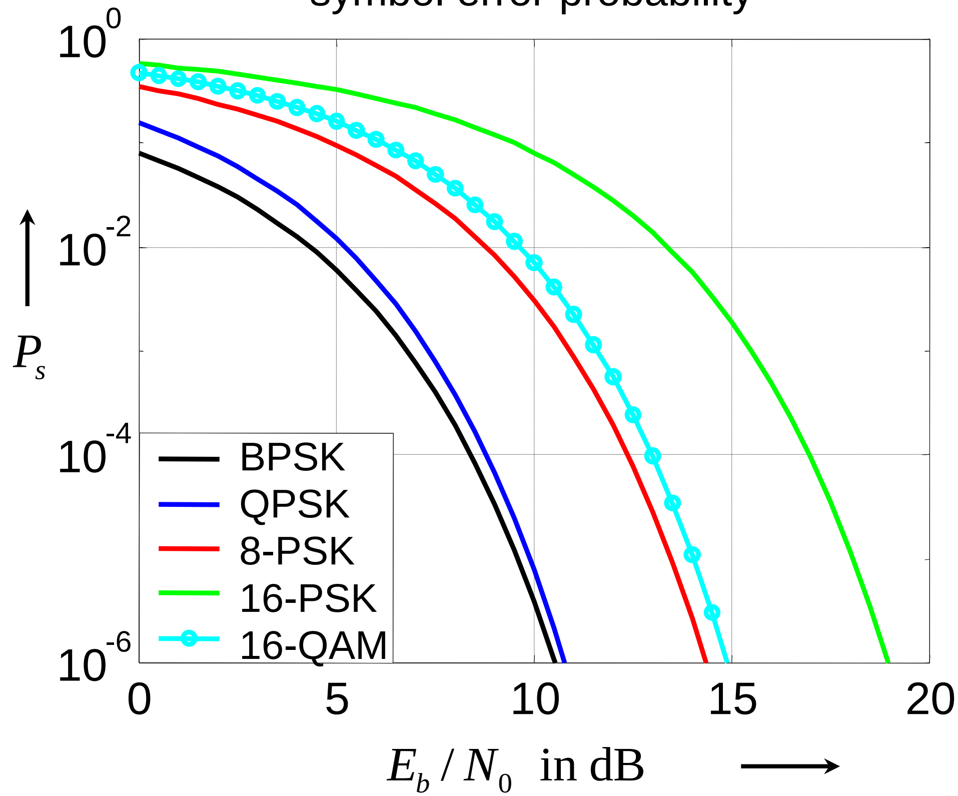
$$P_s^{M\text{-PSK}} \approx \operatorname{erfc}\left(\sin(\pi/M) \cdot \sqrt{\frac{E_s}{N_0}}\right) = \operatorname{erfc}\left(\sin(\pi/M) \cdot \sqrt{m\frac{E_b}{N_0}}\right)$$

Linear Digital Modulation: Symbol Error Probability

$$E_b = \frac{1}{R_c m} E_s$$

(here uncoded transmission)

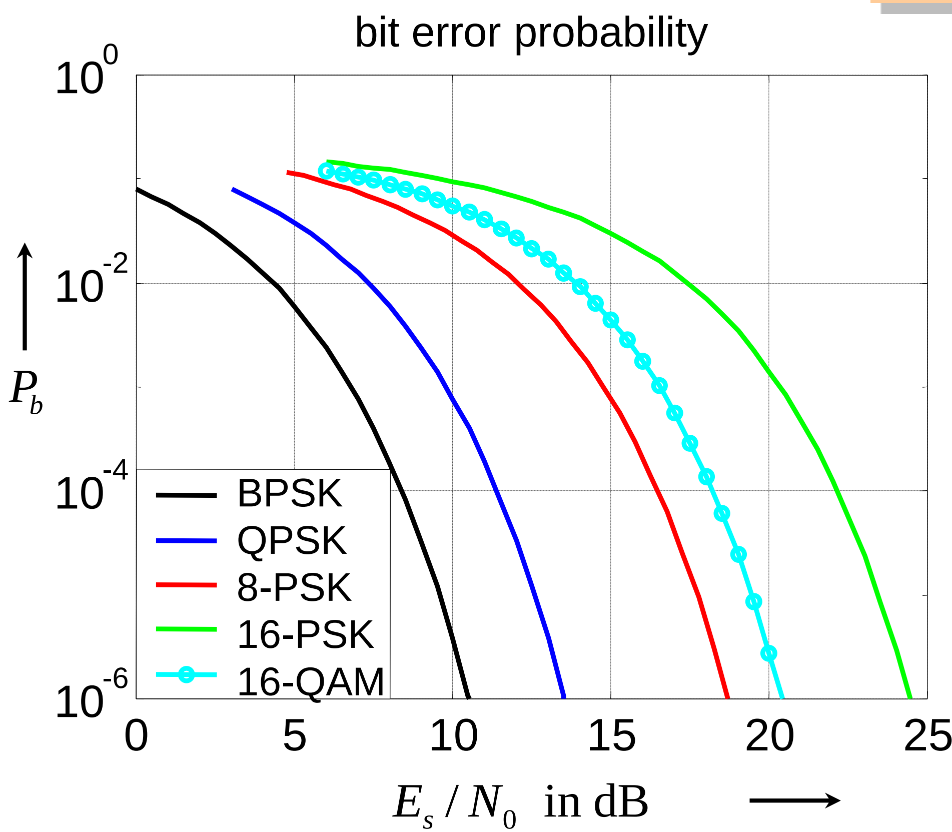


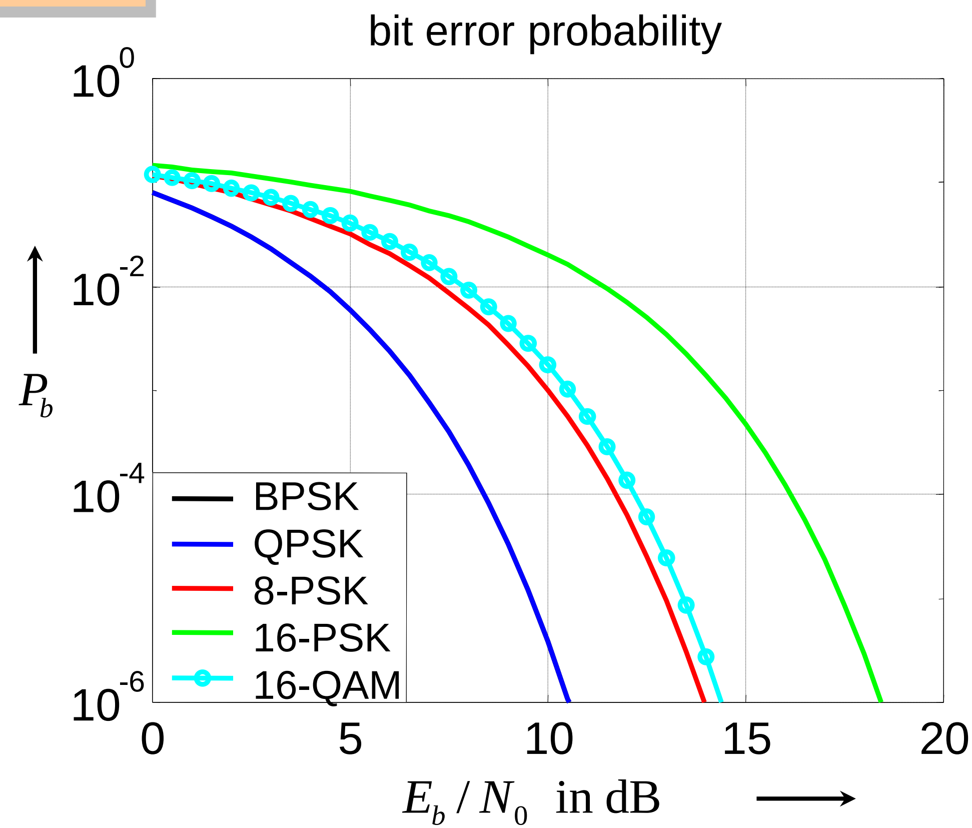


Linear Digital Modulation: Bit Error Probability

$$P_b \approx \frac{1}{m} \cdot P_s$$

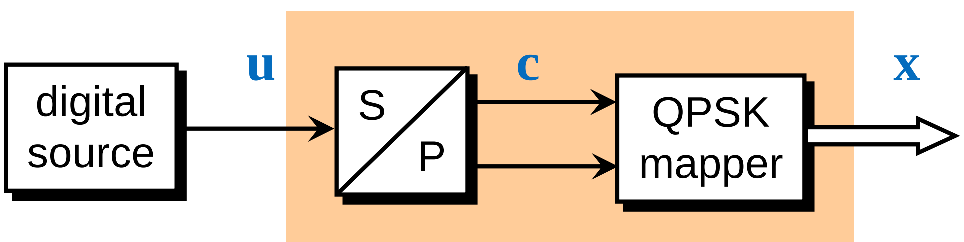
(assuming Gray labeling)





Principle of Coded Modulation (1)

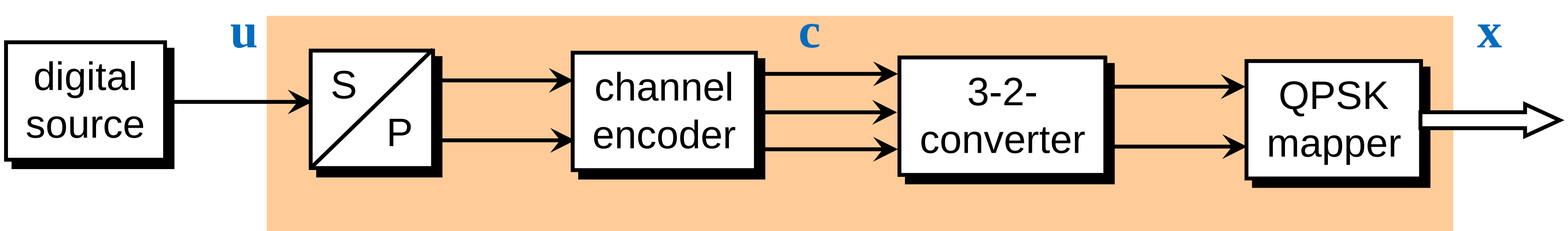
- Increase the number of transmit symbols from M to $\tilde{M}$
  - Instead of m bits we transmit now $\tilde{m}$ bits with each symbol at same bandwidth
  - The additional $\tilde{m} - m$ bits can be generated by a channel code!
- Uncoded QPSK transmission



r_b : information rate
r_c : data rate of code bits
r_s : symbol rate

$r_b = 9.6$ kbit/s $\qquad r_s = r_b / m = 4800$ symbols/s $= 4.8$ kbaud

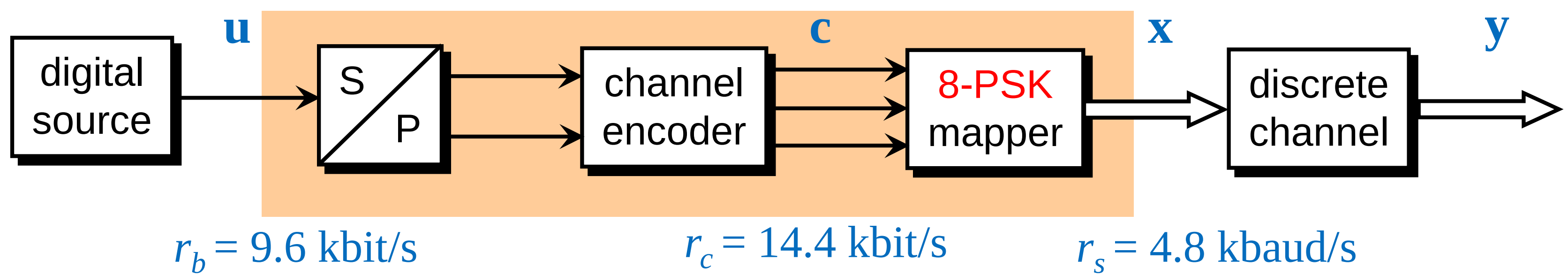
- QPSK transmission with rate 2/3 channel coding



$r_b = 9.6$ kbit/s $\qquad r_c = r_b / R_c = 14.4$ kbit/s $\qquad r_s = r_c / m = 7.2$ kbaud

Principle of Coded Modulation (2)

- Combination of channel encoder and mapper



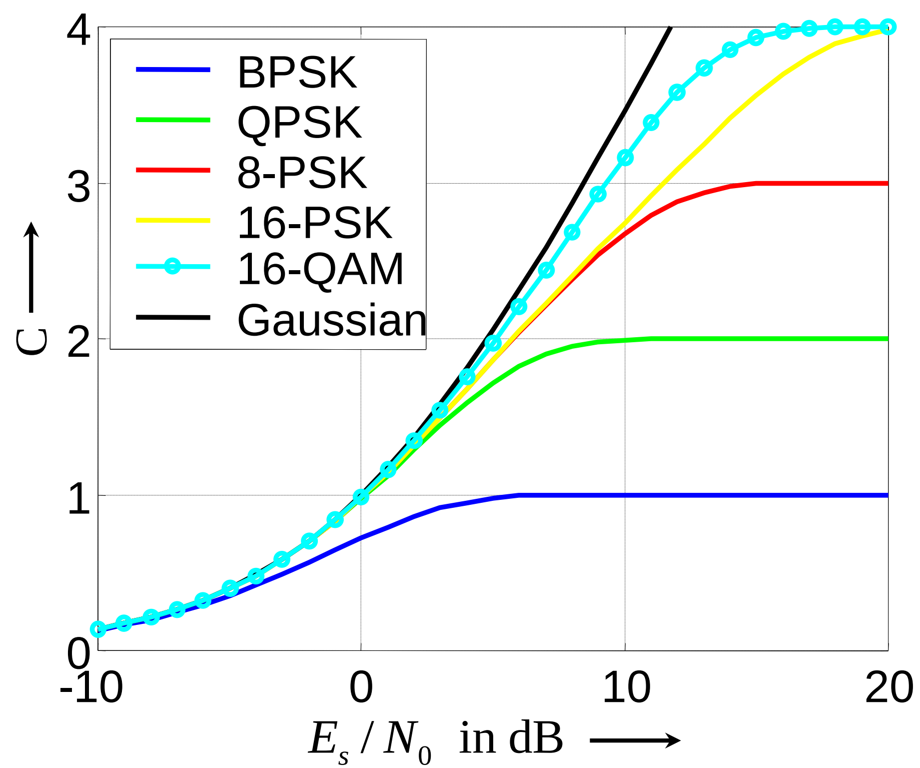
- Channel encoder adds redundancy without increasing bandwidth
- Channel encoder and mapper merge
- Question: How much can we gain from combining channel coding and modulation?
- Example:
  - Convolutional code with $L_c = 7$ and $R_c = 2/3$ gains 6 dB
  - 8-PSK loses roughly 5.3 dB compared to QPSK with respect to E_s/N_0
  - Total gain amount only 6 dB - 5.3 dB = 0.7 dB
  - Is this all???

Capacity of AWGN Channel for Different Linear Digital Modulation Schemes

- Channel capacity for equip. discrete input, continuous output alphabet for AWGN

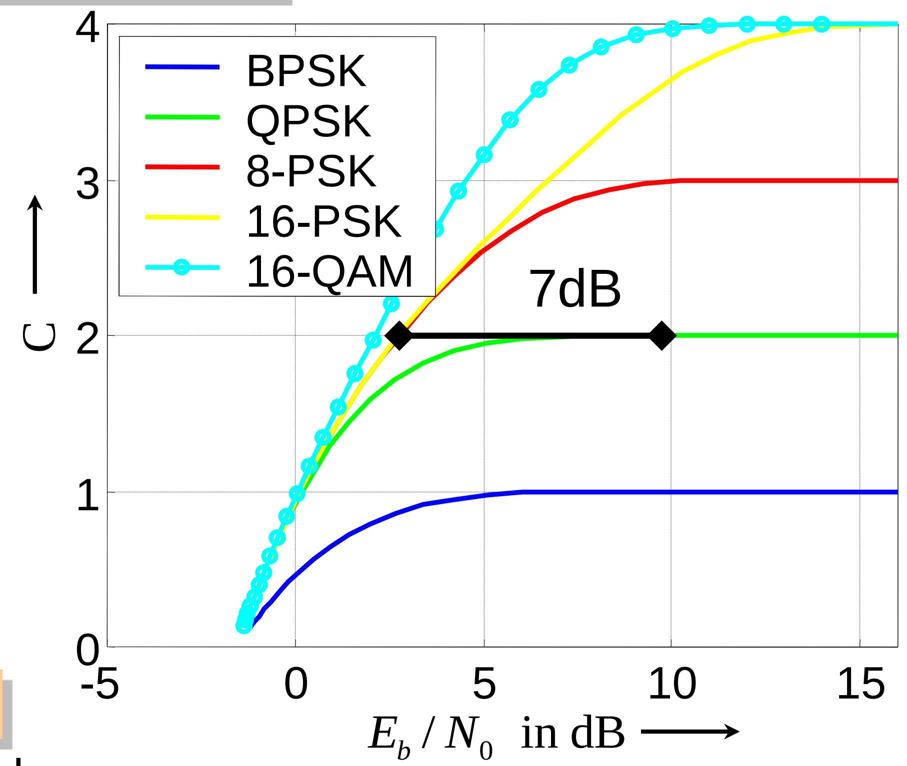
$$C = 2^{-m} \int_{\mathbb{A}_{out}} \sum_{\nu} p_{y|x}\left(\vartheta \middle| x = X_{\nu}\right) \cdot \log_2 \frac{p_{y|x}\left(\vartheta \middle| x = X_{\nu}\right)}{2^{-m} \cdot \sum_{l} p_{y|x}\left(\vartheta \middle| x = X_{l}\right)} d\vartheta$$

- Capacity C vs. E_s/N_0
  - Capacity increases with M
  - For decreasing SNR ($E_s/N_0 \rightarrow -\infty$) capacity tends to zero
  - Asymptotically ($E_s/N_0 \rightarrow \infty$) capacity tends to $m = \mathrm{ld}(M)$, i.e. spectral efficiency η
  - Continuous Gaussian inputs achieve maximum capacity
  - 16-QAM offers higher capacity than 16-PSK as signal space is used more efficiently – asymptotically the same spectral efficiency $\eta = 4$ bit/s/Hz is achieved of course



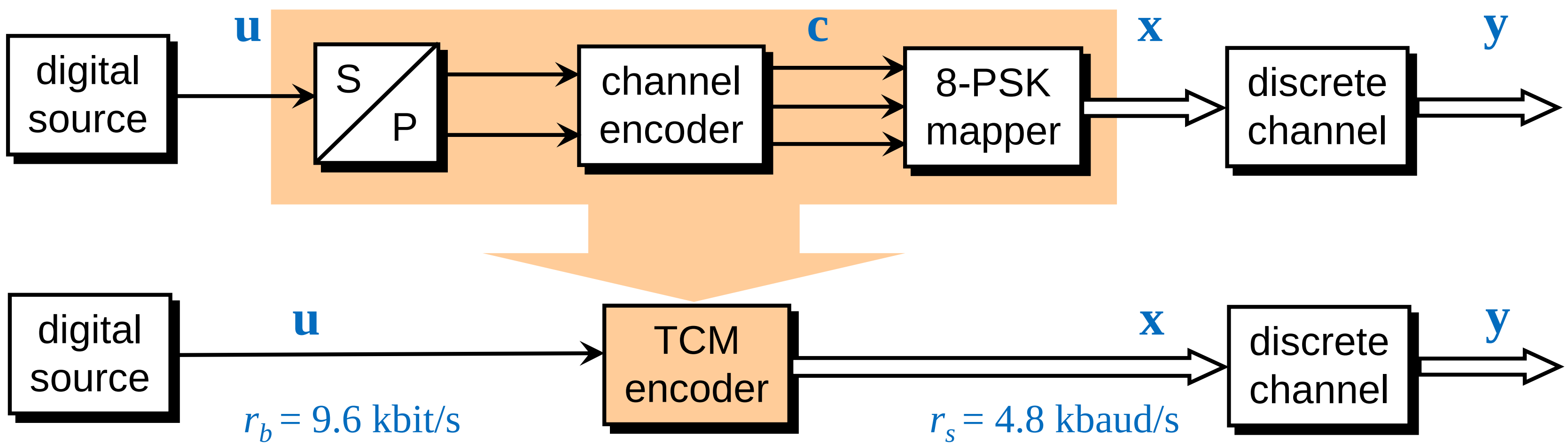
Capacity of AWGN Channel for Different Linear Digital Modulation Schemes

- Capacity C vs. E_b/N_0
  - $p(\mathbf{y}|\mathbf{x})$ depends on E_s/N_0 → $C = f\left(E_s/N_0\right) = f\left(R_c \cdot E_b/N_0\right)$ implicit equation for $R_c = C$
  - No error-free communication possible for $E_b/N_0 <$ -1.59 dB
  - For large SNR the capacity of all schemes equals corresponding η
- Comparison: (e.g. $\eta = 2$ bit/s/Hz)
  - Error-free transmission with uncoded QPSK requires $E_b/N_0 > 9.5$ dB
  - Rate 2/3 coded 8-PSK needs only $E_b/N_0 > 2.5$ dB → gain of 7dB
- Doubling the modulation size is sufficient
  - Doubling size $M \rightarrow \tilde{M} = 2M \quad m \rightarrow \tilde{m} = m+1$
  - Codes of rate $R_c = m/(m+1) = k/(k+1)$ are used



Principle of Coded Modulation (3)

- Combination of channel encoder and mapper



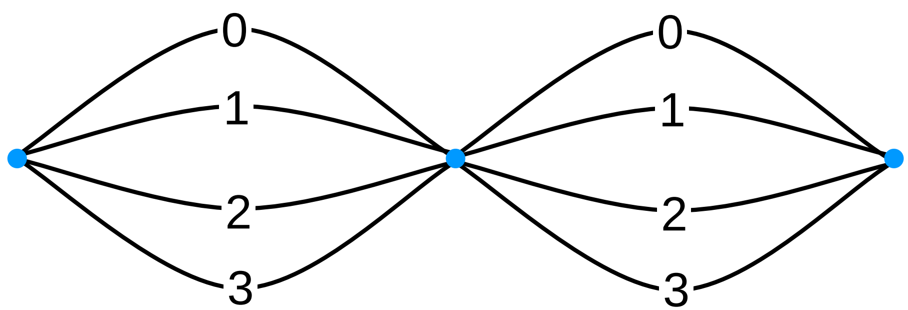
- Now detection of symbol sequences
- Maximum likelihood approach: $\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\| \mathbf{y} - \mathbf{x} \right\|^2$
- **Minimum squared Euclidean distance** should be maximized (for AWGNC)

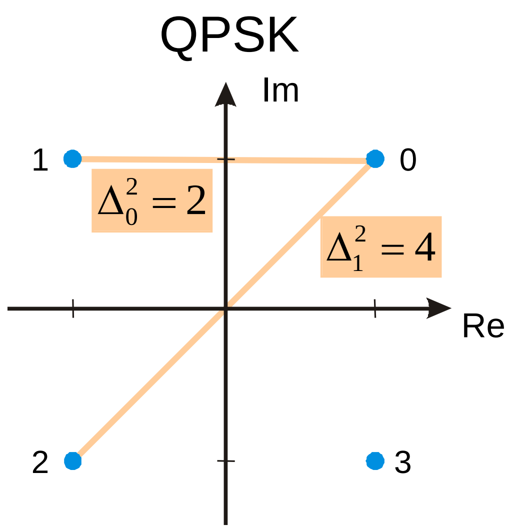
$$\Delta_f^2 = \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} d_e^2\left(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\right) = \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} \left\| \mathbf{x}^{(1)} - \mathbf{x}^{(2)} \right\|^2$$

For pure FEC the minimum Hamming distance / free distance should be maximized

First Approaches Towards Trelliscoded Modulation

- Uncoded QPSK ($\eta = 2$ bit/s/Hz)
  - All symbol sequences are possible





- Minimum squared Euclidean distance between 2 sequences

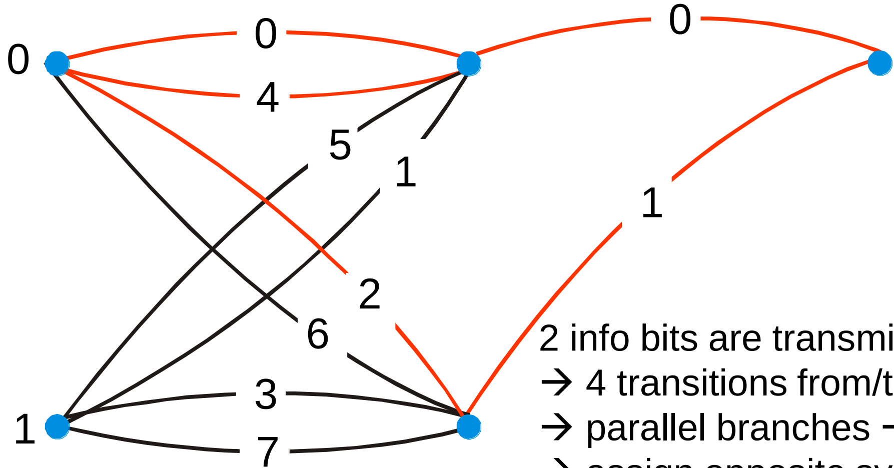
$$\Delta_f^2 = \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} d_e^2\left(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\right) = \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} \left\|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\right\|^2 = \Delta_0^2 = 2$$

is determined by minimum distance between 2 QPSK symbols:

$$\Delta_0^2(\text{QPSK}) = \left(2 \cdot \frac{1}{\sqrt{2}}\right)^2 = 2$$

First Approaches Towards Trelliscoded Modulation

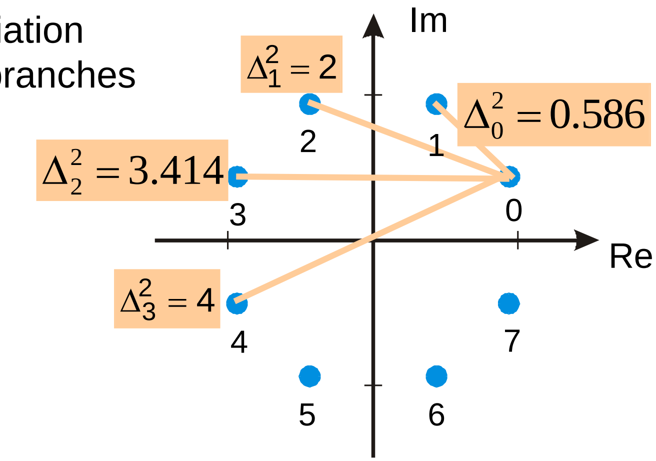
- Trelliscoded 8-PSK (1 memory → 2 states)



2 info bits are transmitted with each symbol
→ 4 transitions from/to each state
→ parallel branches → shortest deviation
→ assign opposite symbols to par. branches

- Parallel branches: $d_{ep}^2 = \Delta_3^2 = 4$
- Non-parallel branches:
  $$d_{ef}^2 = d_e^2(0,2) + d_e^2(0,1) = 2 + 0.586 = 2.586$$
- Minimum squared Euclidean distance
  $$\Delta_f^2 = \min\left(d_{ep}^2, d_{ef}^2\right) = 2.586$$

8-PSK

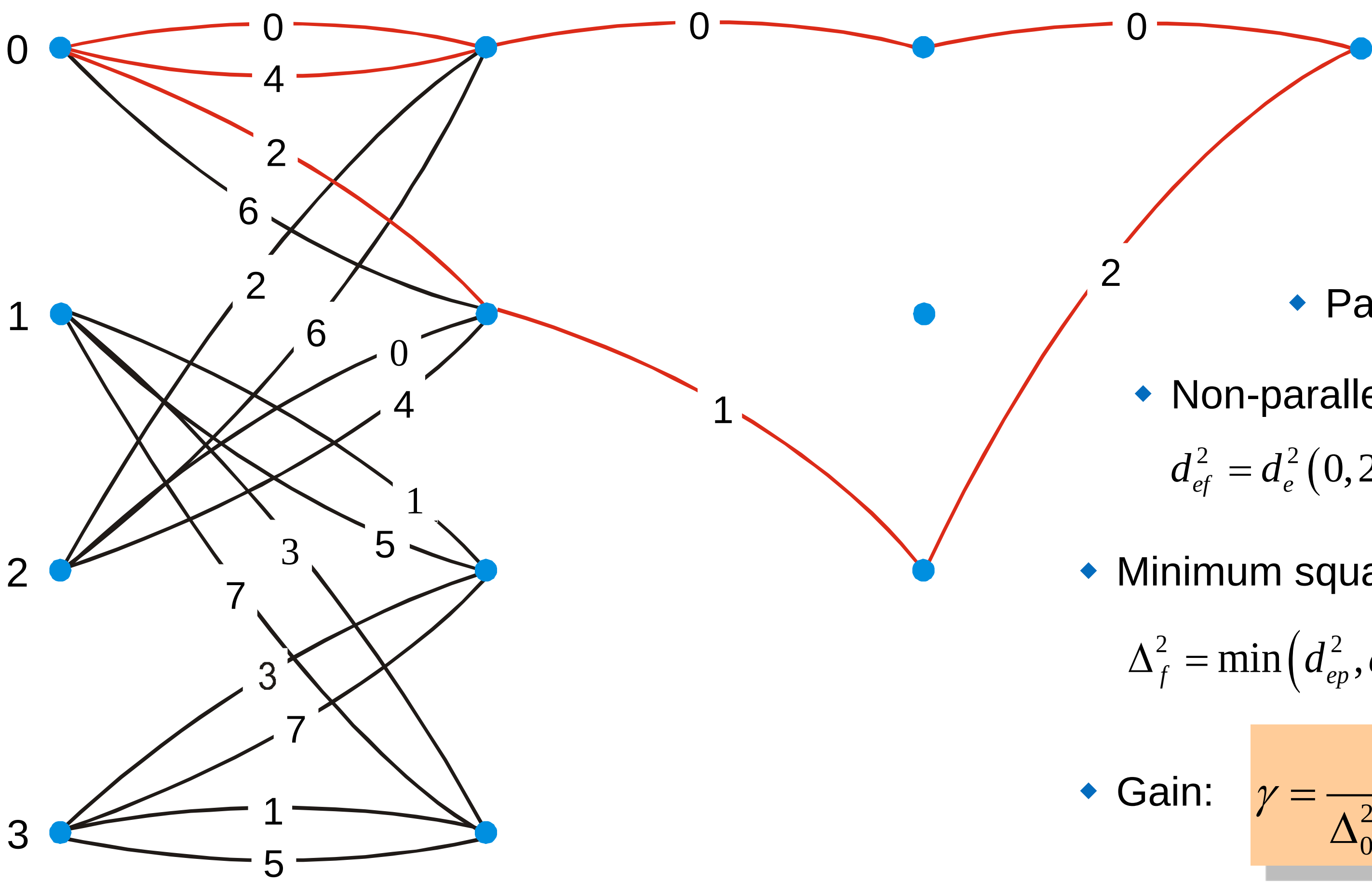


- **Gain**: Ratio of minimum squared Euclidian distance of coded and uncoded sequences

$$\gamma = \frac{\Delta_f^2}{\Delta_0^2(\text{QPSK})} = \frac{2.586}{2} = 1.293 \mathrel{\hat{=}} 1.12 \text{ dB}$$

First Approaches Towards Trelliscoded Modulation

- Trelliscoded 8-PSK (4 states)



- Parallel branches: $d_{ep}^2 = \Delta_3^2 = 4$
- Non-parallel branches:

$$d_{ef}^2 = d_e^2(0,2) + d_e^2(0,1) + d_e^2(0,2) = 4.586$$

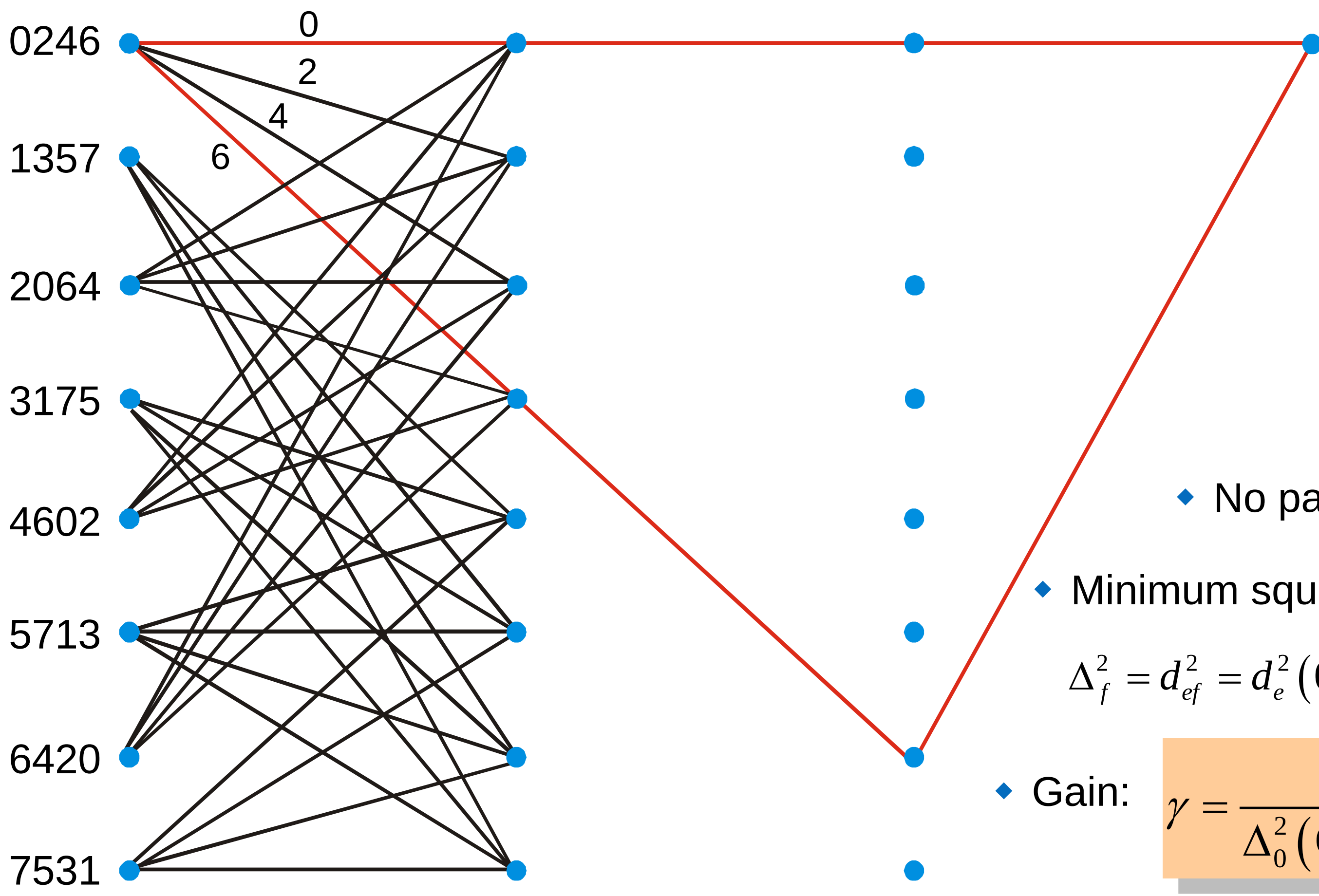
- Minimum squared Euclidean distance

$$\Delta_f^2 = \min\left(d_{ep}^2, d_{ef}^2\right) = 4$$

- Gain: $$\gamma = \frac{\Delta_f^2}{\Delta_0^2(\mathrm{QPSK})} = \frac{4}{2} = 2 \mathrel{\hat{=}} 3\ \text{dB}$$

First Approaches Towards Trelliscoded Modulation

- Trelliscoded 8-PSK (8 states)



first symbol at each node corresponds to first branch, second symbol to second branch, …

- No parallel branches
- Minimum squared Euclidean distance

$$\Delta_f^2 = d_{ef}^2 = d_e^2(0,6) + d_e^2(0,7) + d_e^2(0,6) = 4.586$$

- Gain: $\gamma = \dfrac{\Delta_f^2}{\Delta_0^2(\text{QPSK})} = 2.293 \mathrel{\hat{=}} 3.6 \text{ dB}$

Some Remarks about TCM

- Inserting memory leads to significant gains
  - Not all symbol combinations can occur in a sequence
    → increases distance between sequences

- Increasing the number of states leads to an improved performance, but also to a larger complexity for the decoder

- Calculated gain is realized only asymptotically (large SNR)

- Question:
  - Is there a systematic way to construct optimum TCM-Codes?
- Answer:
  - No. Optimum structures have been found for AWGN by computer search.
  - But, there are some heuristic rules that help us to find good codes (without any guarantee to find the best code)

Ungerböck's Set-Partitioning

- Mapping by set partitioning
  - Aim: optimizing the distance properties of TCM codes
  - Parallel branches should be assigned to symbols with large Euclidian distance
  - Common branches are separated by the trellis structure
    → can be assigned to symbols with smaller Euclidian distance
- **Strategy for a successive separation of the signal space**
  - Start with complete signal space $\mathbf{B} = \mathcal{A}_{\text{in}}$
  - Separate $\mathbf{B}$ into 2 subsets $\mathbf{B}_0^{(1)}$ and $\mathbf{B}_1^{(1)}$ so that the Euclidian distances between the symbols of one subset is increased:

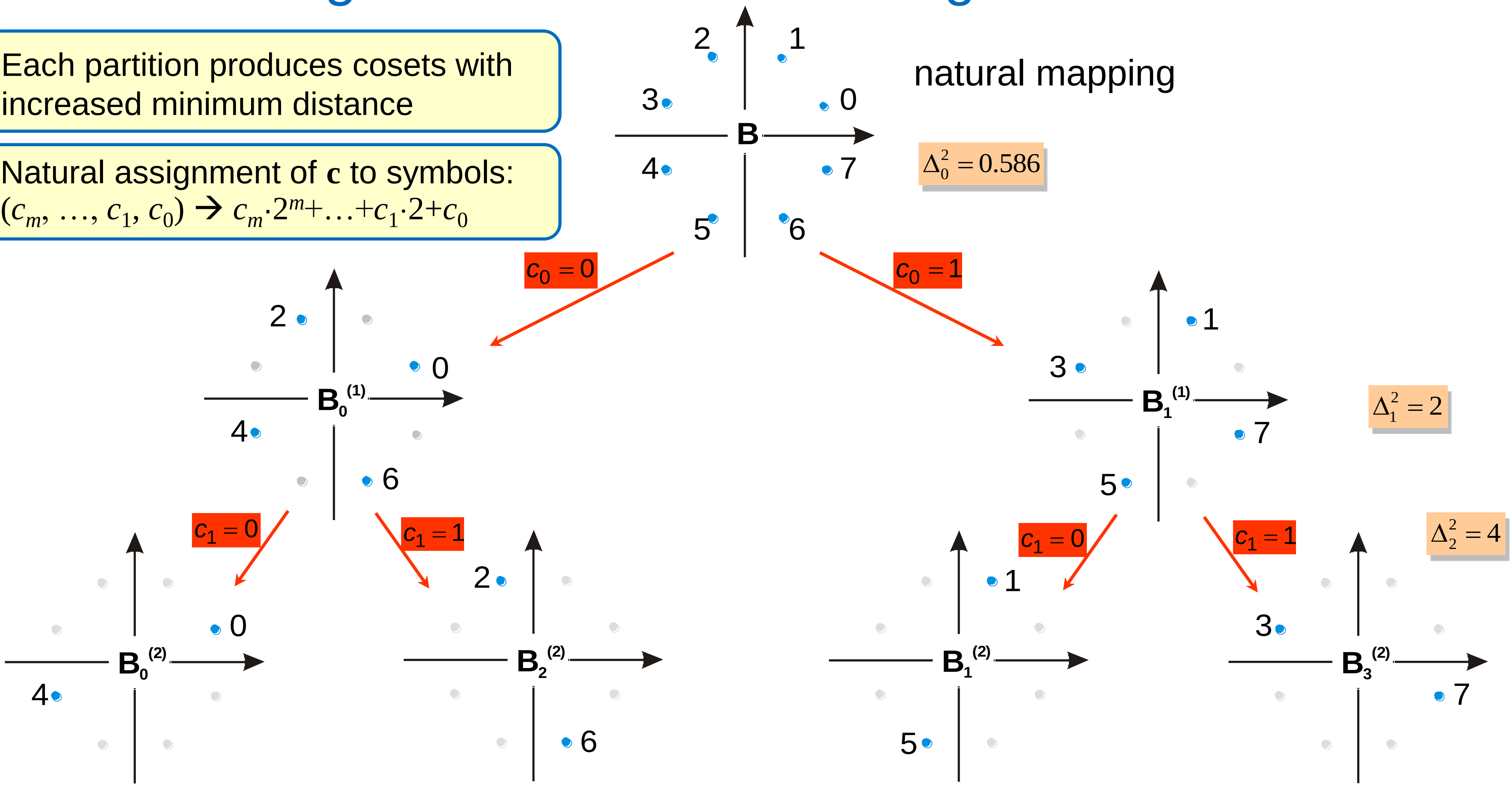
$$\mathbf{B} \rightarrow \left\{\mathbf{B}_0^{(1)}, \mathbf{B}_1^{(1)}\right\}$$

  - Repeat separation of $\mathbf{B}_0^{(1)} \rightarrow \left\{\mathbf{B}_0^{(2)}, \mathbf{B}_2^{(2)}\right\}$ and $\mathbf{B}_1^{(1)} \rightarrow \left\{\mathbf{B}_1^{(2)}, \mathbf{B}_3^{(2)}\right\}$ to increase the distance within the subsets
  - Repeat separation of all generated subsets until the subsets contain only one symbol → $\tilde{m} = m + 1$ partitioning steps

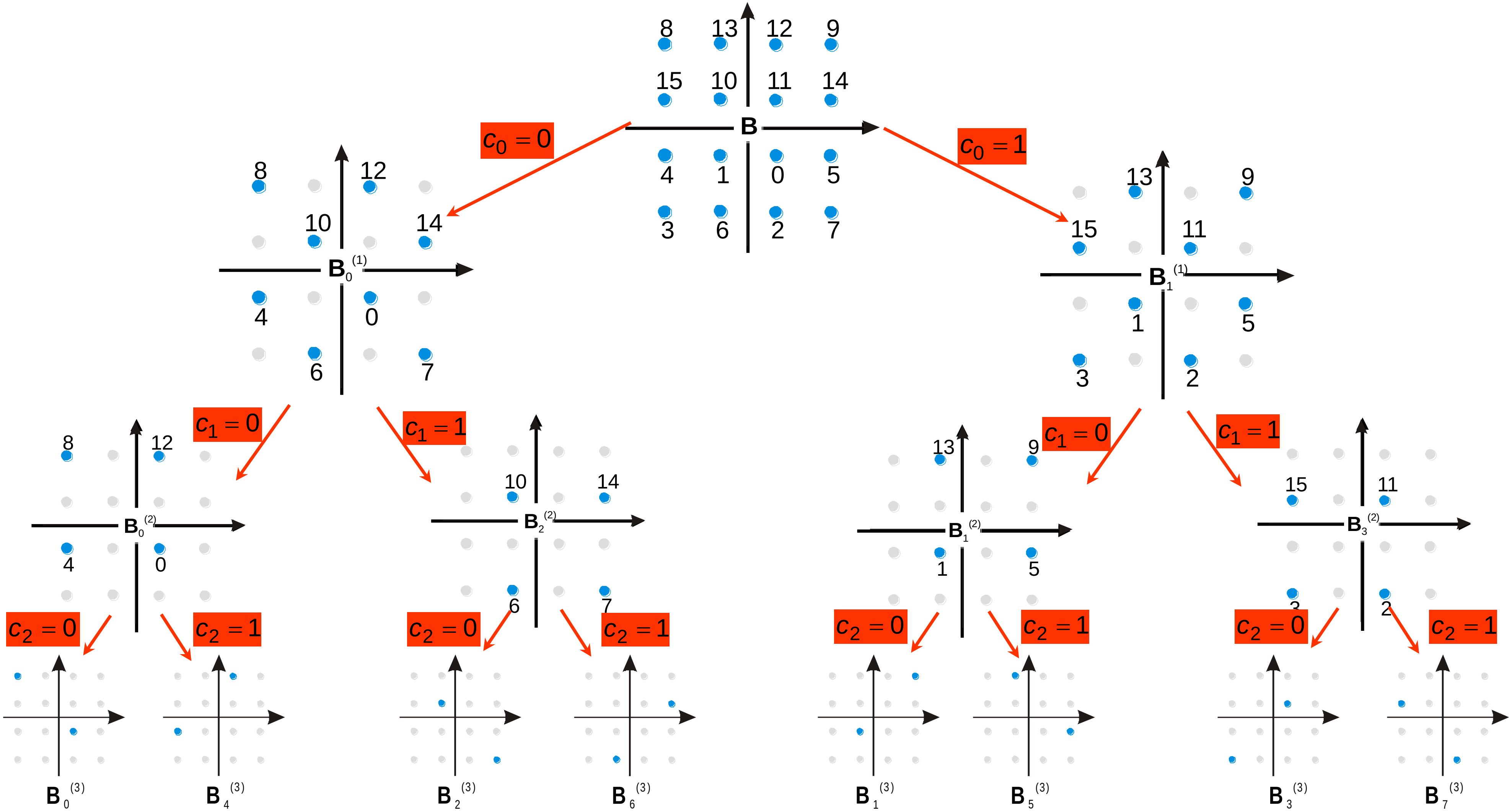
Ungerböck's Set-Partitioning for 8-PSK

Each partition produces cosets with increased minimum distance

Natural assignment of **c** to symbols:
$(c_m, \ldots, c_1, c_0) \rightarrow c_m \cdot 2^m + \ldots + c_1 \cdot 2 + c_0$

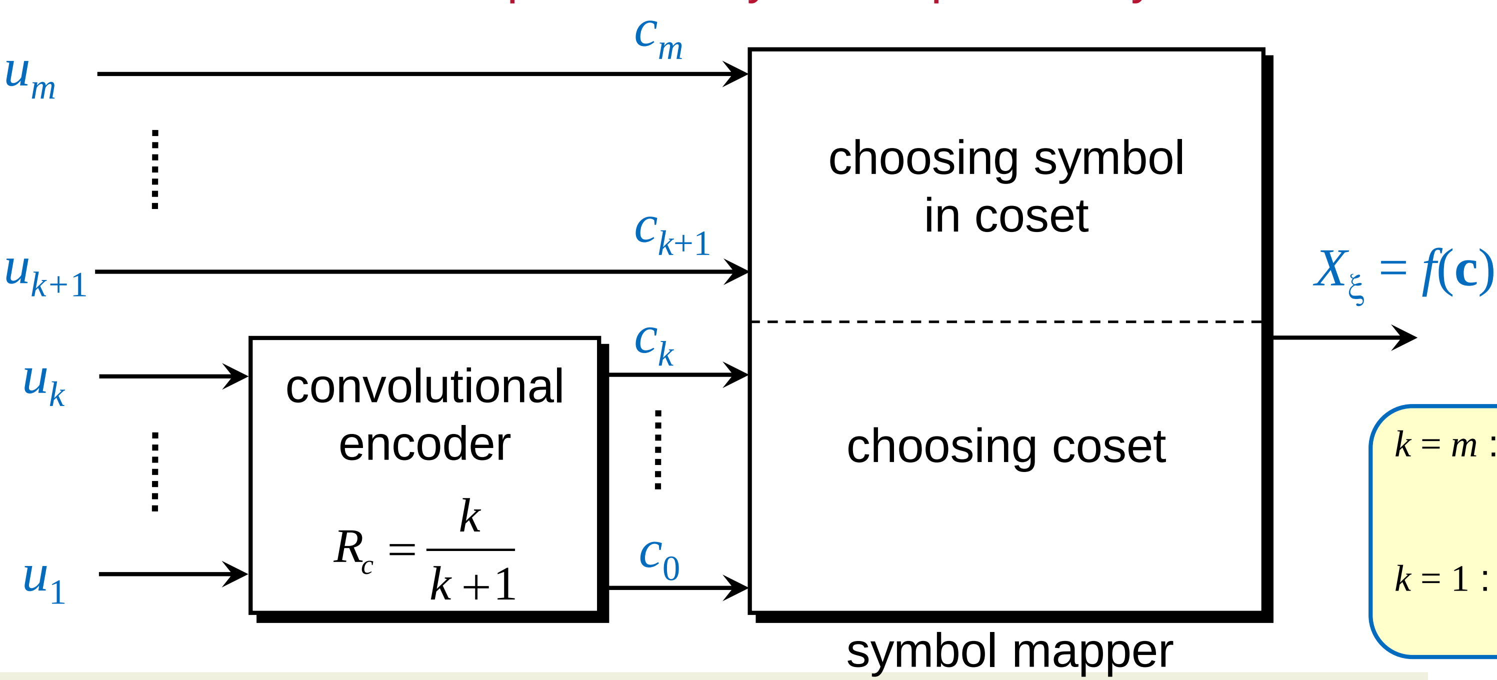


Ungerböck's Set-Partitioning for 16-QAM



Principle Structure of TCM Encoders

- Spectral efficiency of TCM with $M = 2^{m+1}$: m bits/s/Hz
  - Bits $u_1 \ldots u_k$ are convolutionally encoded
    → encoded bits $c_0 \ldots c_k$ determine symbol coset
  - Bits $u_{k+1} \ldots u_m$ remain uncoded and determine symbol within coset
  - Weak uncoded bits are protected by well separated symbols in cosets



$k = m$: $R_c = m/(m+1)$
all info bits are encoded
$k = 1$: $R_c = 1/2$
m-1 uncoded bits

Ungerböck's Set-Partitioning

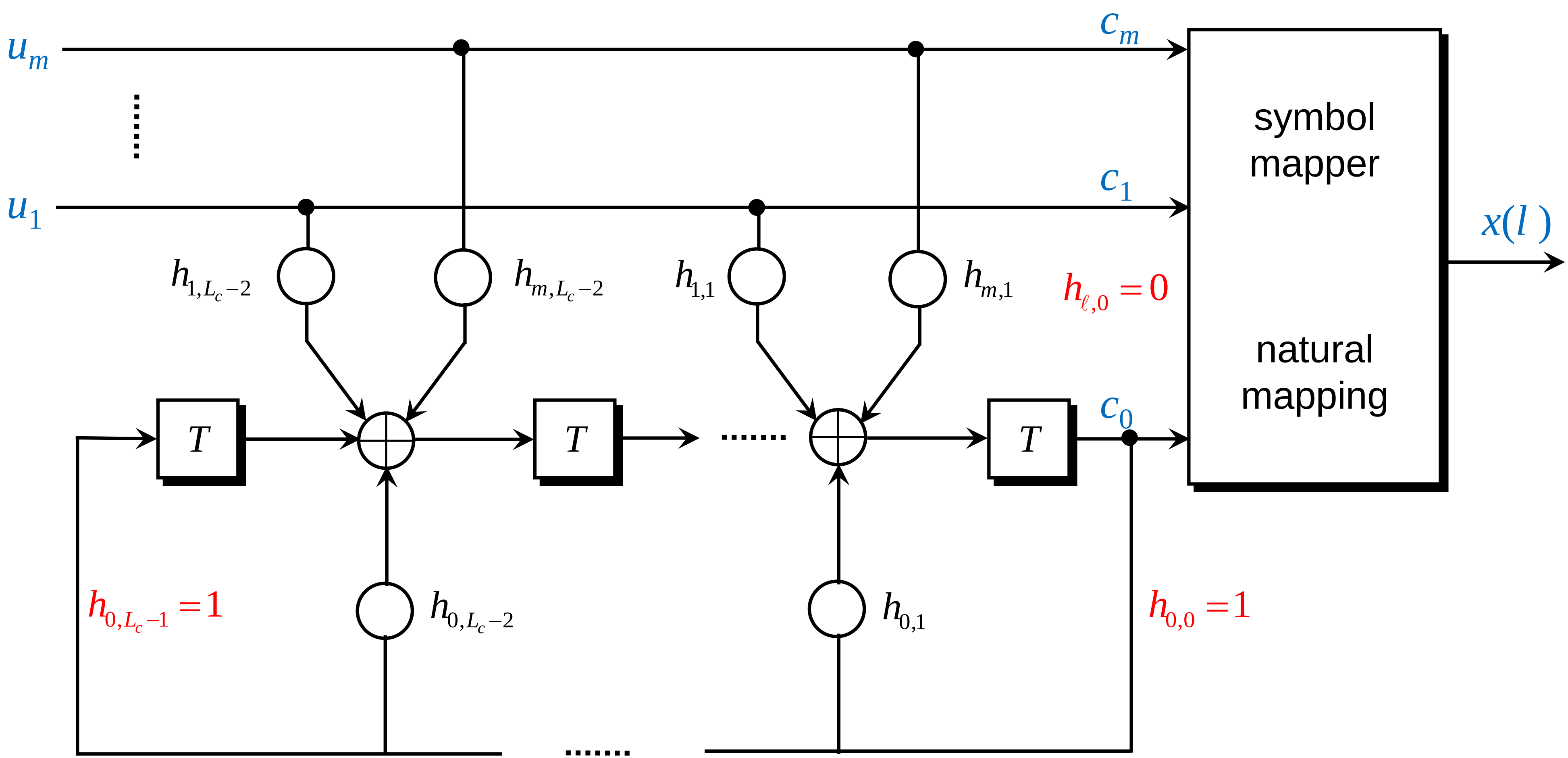
- For AWGN the code construction should maximize the minimum Euclidian distance between sequences
  → combined optimization of convolutional code and mapping

- Guidelines for code construction of Ungerböck
  - If there are uncoded bits, they should be assigned to the last partitioning steps, i.e. they determine a symbol of a certain subset!
    $(u_{k+1}, \ldots, u_m)$ determine symbol within cosets of partitioning step m-k
  - Branches arriving at the same state or leaving the same state should be assigned to symbols of the same subset!
  - All symbols should occur equally likely!

- These guidelines do not lead to unique optimum codes, but they reduce the space to search in!

- In practice recursive systematic codes (RSC) are used mostly

Systematic TCM Encoder with Recursive Shift Register



Optimal Codes of Ungerböck for 8-PSK

number of states	k	h_0	h_1	h_2	Δ_f^2	$G_{8\text{-PSK/QPSK}}$ [dB]	gain at $P_b = 10^{-5}$ [dB]
4	1	5	2		4.000	3.01	2.4
8	2	11	02	04	4.586	3.60	2.8
16	2	23	04	16	5.172	4.13	3.0
32	2	45	16	34	5.758	4.59	3.3
64	2	103	030	066	6.343	5.01	3.6
128	2	277	054	122	6.586	5.17	
256	2	435	072	130	7.515	5.75	

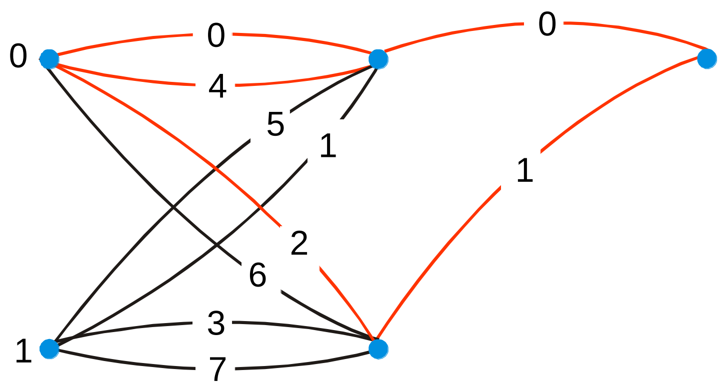
octal representation of coefficients h_i

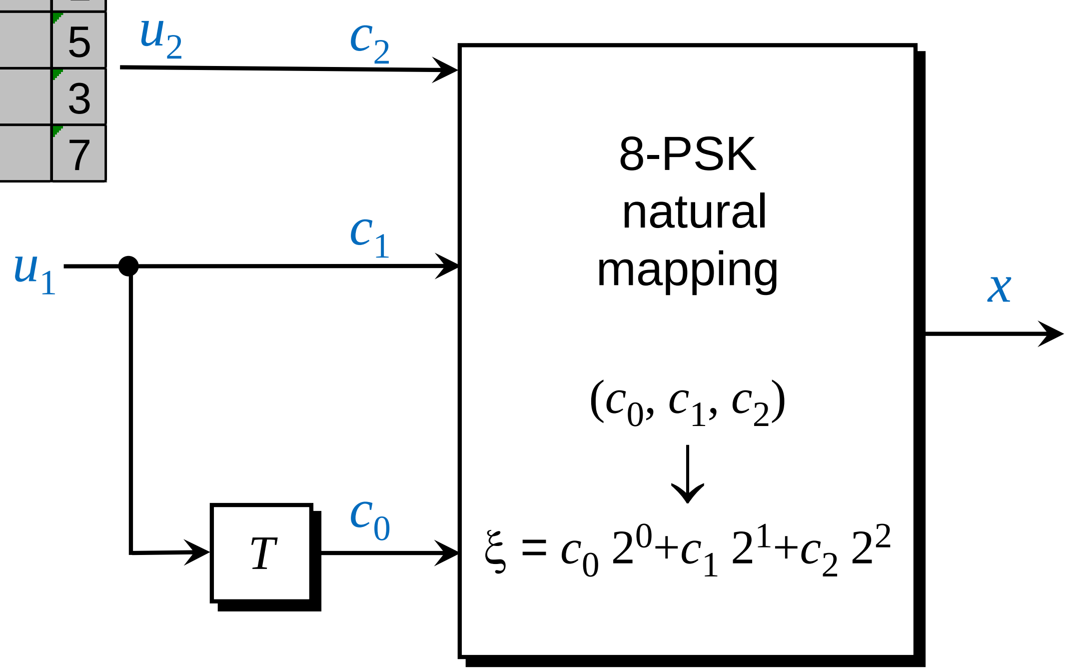
- Parallel branches dominate for 4 states ($k = 1$)
- For more than 4 states no parallel branches occur anymore ($k = 2$)
- Minimum squared Euclidean distance (→ gain) increases with number of states

Optimal Codes of Ungerböck: 2 states, 8-PSK

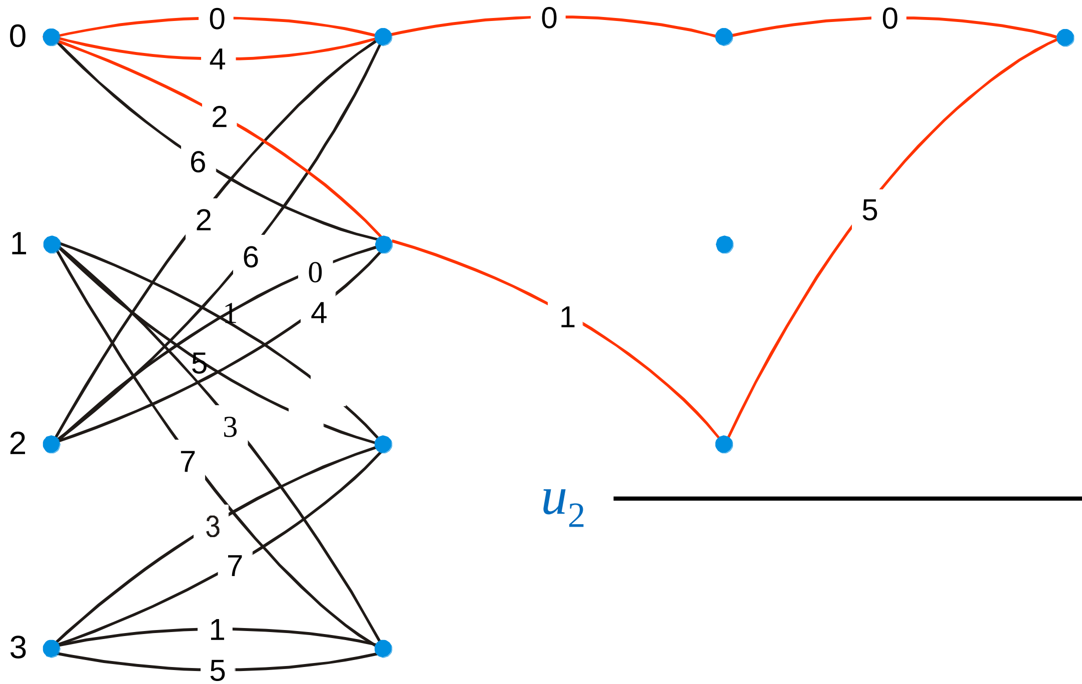
(u_1,u_2)	state	successive state	(c_0,c_1,c_2)	x
00	0	0	000	0
01	0	0	001	4
10	0	1	010	2
11	0	1	011	6
00	1	0	100	1
01	1	0	101	5
10	1	1	110	3
11	1	1	111	7

- parallel branches are assigned to opposite symbols → (0,4), (2,6)
- which of these symbols is transmitted is determined by the uncoded bit c_2
- Coded bits c_0 and c_1 determine the symbol subset



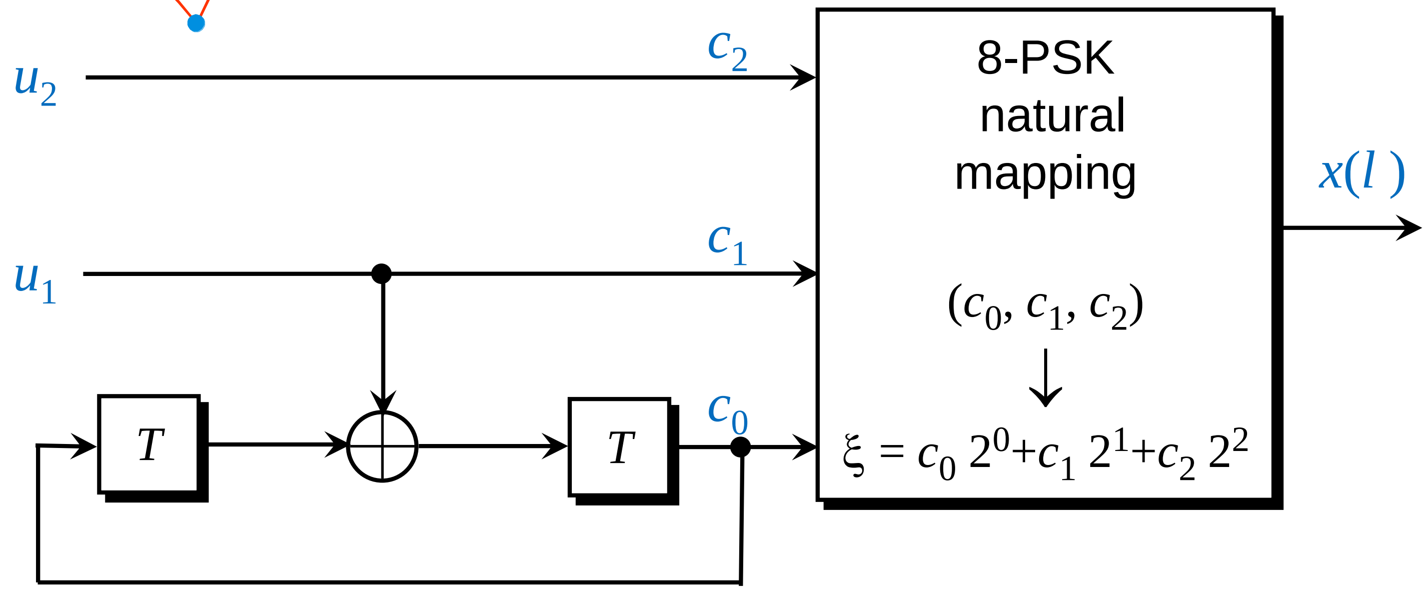


Optimal Codes of Ungerböck: 4 states, 8-PSK



$$\mathbf{h}_0 = \begin{pmatrix} h_{0,2} & h_{0,1} & h_{0,0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} = 5$$

$$\mathbf{h}_1 = \begin{pmatrix} h_{1,2} & h_{1,1} & h_{1,0} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = 2$$

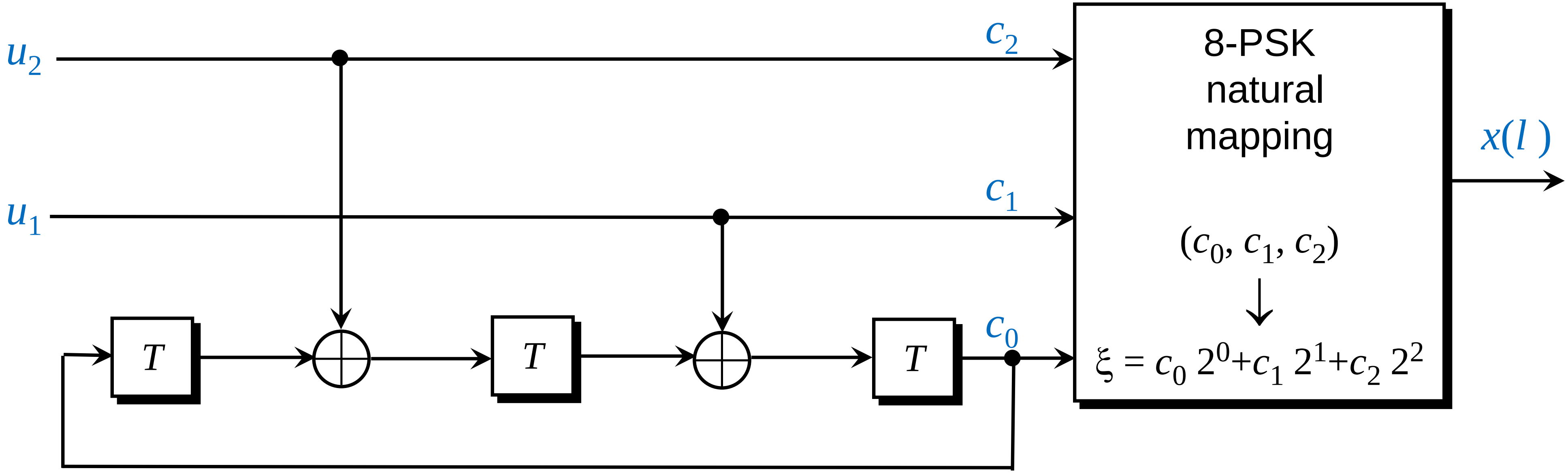


Optimal Codes of Ungerböck: 8 states, 8-PSK

$$\mathbf{h}_0 = \begin{pmatrix} h_{0,3} & h_{0,2} & h_{0,1} & h_{0,0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = 11$$

$$\mathbf{h}_1 = \begin{pmatrix} h_{1,3} & h_{1,2} & h_{1,1} & h_{1,0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} = 02$$

$$\mathbf{h}_2 = \begin{pmatrix} h_{2,3} & h_{2,2} & h_{2,1} & h_{2,0} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} = 04$$



Optimal Codes of Ungerböck for 16-PSK

number of states	k	h_0	h_1	h_2	Δ_f^2	$G_{16\text{-PSK}/8\text{-PSK}}$ [dB]	gain at $P_b = 10^{-5}$ [dB]
4	1	5	2		1.324	3.54	2.3
8	1	13	04		1.476	4.01	2.7
16	1	23	04		1.628	4.44	2.9
32	1	45	10		1.910	5.13	3.2
64	1	103	024		2.000	5.33	3.5
128	1	203	024		2.000	5.33	
256	2	427	176	374	2.085	5.51	

- 4 parallel branches exist up to 128 states ($k = 1$)
  - Nonparallel branches dominate up to 32 states; smaller distance than parallel branches!
  - Parallel branches dominate for 64 and 128 states ($k = 1$)
- For more than 128 states, only 2 parallel branches exist ($k = 2$)

ML-Decoding with Viterbi-Algorithm

- Maximum-Likelihood Decoding: Determine that symbol sequence $\hat{\mathbf{x}}$ with minimum Euclidian distance to the received sequence $\mathbf{y}$

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{y}-\mathbf{x}\|^2$$

demodulation and decoding are no longer separated
→ joint demodulation and decoding → TCM decoding

- Efficient realization for ML Decoding is given by the **Viterbi-Algorithm**
  - The difference in contrast to decoding of convolutional codes is given by the **metric**
  - If parallel branches occur, only the best branch is considered
- Squared Euclidian distance

$$d_e^2(\mathbf{x},\mathbf{y}) = \sum_{\ell}\big(y(\ell)-x(\ell)\big)\cdot\big(y(\ell)-x(\ell)\big)^* = \sum_{\ell}\left(|y(\ell)|^2 - x(\ell)\cdot y(\ell)^* - x(\ell)^* y(\ell) + |x(\ell)|^2\right)$$
$$= \sum_{\ell}\left(|x(\ell)|^2 + |y(\ell)|^2 - 2\cdot\mathrm{Re}\left\{x(\ell)^* y(\ell)\right\}\right)$$

  - For PSK modulation $\mu(\mathbf{x},\mathbf{y}) = \sum_{\ell}\mathrm{Re}\left\{x(\ell)^*\cdot y(\ell)\right\}$ ➡ $\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \mu(\mathbf{x},\mathbf{y})$

Analytical Approximation of Bit Error Probability

- Recall: Calculation of bit error rate of convolutional codes by distance spectrum
  - Distance spectrum

$$T(W,D,L) = \sum_w \sum_d \sum_l T_{w,d,l} \cdot W^w D^d L^l$$

L = sequence length

W = weight of uncoded input sequence

D = weight of coded output sequence

  - Union bound for bit error rate

$$P_b \le \sum_d c_d \cdot P_d = \frac{1}{2} \cdot \sum_d c_d \cdot \operatorname{erfc}\left(\sqrt{d \cdot R_c \frac{E_b}{N_0}}\right)$$

  - Number of non-zero info bits for all sequences with Hamming weight d

$$c_d = \sum_w \sum_\ell w \cdot T_{w,d,\ell}$$

- Notice: Convolutional codes are linear, whereas **TCM are nonlinear** due to the mapping of vector $\mathbf{c}$ to transmit symbols x
  - Comparison of all sequences with all-zero sequence is not sufficient
  - All pairs of sequences have to be considered → larger effort

Analytical Approximation of Bit Error Probability

- Assumption: Optimal maximum likelihood decoding by Viterbi algorithm
- Error probability of sequence $\mathbf{x}$:

$$
\begin{aligned}
P_e(\mathbf{x}) &= P\big(\mathbf{y} \notin \mathrm{D}(\mathbf{x})\big) \\
&= P\left(\mathbf{y} \in \bigcup_{\mathbf{x}' \in \Gamma} \bar{\mathrm{D}}(\mathbf{x},\mathbf{x}')\right) \\
&\leq \sum_{\mathbf{x}' \in \Gamma} P\big(\mathbf{y} \in \bar{\mathrm{D}}(\mathbf{x},\mathbf{x}')\big) \\
&= \sum_{\mathbf{x}' \in \Gamma} P\big(\mathbf{x} \rightarrow \mathbf{x}'\big)
\end{aligned}
$$

with $\mathrm{D}(\mathbf{x}) = \left\{\mathbf{y} \middle| P(\mathbf{y} \mid \mathbf{x}) > P(\mathbf{y} \mid \mathbf{x}'),\ \forall \mathbf{x}' \in \Gamma\right\}$

with $\bar{\mathrm{D}}(\mathbf{x},\mathbf{x}') = \left\{\mathbf{y} \middle| P(\mathbf{y} \mid \mathbf{x}) < P(\mathbf{y} \mid \mathbf{x}')\right\}$

equality “=“ holds for disjoint sets $\bar{\mathrm{D}}(\mathbf{x},\mathbf{x}')$

- **Pairwise Error Probability** (**PEP**):

$$
P\big(\mathbf{y} \in \bar{\mathrm{M}}(\mathbf{x},\mathbf{x}')\big) = P(\mathbf{x} \rightarrow \mathbf{x}') = P\left(\|\mathbf{y}-\mathbf{x}\|^2 > \|\mathbf{y}-\mathbf{x}'\|^2\right) = \frac{1}{2} \cdot \mathrm{erfc}\left(\sqrt{\frac{\|\mathbf{x}-\mathbf{x}'\|^2}{4N_0/T_s}}\right)
$$

Analytical Approximation of Bit Error Probability

- Pairwise error probability:

$$P(\mathbf{x} \to \mathbf{x}') = \frac{1}{2} \cdot \operatorname{erfc}\left(\sqrt{\frac{\|\mathbf{x}-\mathbf{x}'\|^2}{4N_0/T_s}} \right) = \frac{1}{2} \cdot \operatorname{erfc}\left(\sqrt{d_e^2(\mathbf{x},\mathbf{x}') \frac{E_s/T_s}{4N_0/T_s}} \right)$$

with

$$d_e^2(\mathbf{x},\mathbf{x}') = \frac{\|\mathbf{x}-\mathbf{x}'\|^2}{E_s/T_s}$$

- Error Probability of sequence $\mathbf{x}$:

$$P_e(\mathbf{x}) \le \sum_{\mathbf{x}'\in\Gamma} P(\mathbf{x} \to \mathbf{x}') = \frac{1}{2} \cdot \sum_{\mathbf{x}'\in\Gamma} \operatorname{erfc}\left(\sqrt{d_e^2(\mathbf{x},\mathbf{x}') \frac{E_s}{4N_0}} \right)$$

  - Not only Hamming distance (number of different symbols in $\mathbf{x}$ and $\mathbf{x}'$), but the Euclidian distance between the symbols are of importance

- Simplification of the probability by using the approximation

$$\operatorname{erfc}\left(\sqrt{a+b}\right) \le \operatorname{erfc}\left(\sqrt{a}\right) \cdot e^{-b}$$

$$a = \Delta_f^2 \cdot \frac{E_s}{4N_0} \qquad b = d_e^2(\mathbf{x},\mathbf{x}') \cdot \frac{E_s}{4N_0} - \Delta_f^2 \cdot \frac{E_s}{4N_0}$$

Analytical Approximation of Bit Error Probability

$$\Delta_f^2 = \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} d_e^2\left(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\right) = \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} \left\|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\right\|^2$$

- Error Probability of sequence $\mathbf{x}$ is bounded by:

$$P_e(\mathbf{x}) \le \frac{1}{2} \cdot \sum_{\mathbf{x}' \in \Gamma} \operatorname{erfc}\left(\sqrt{\Delta_f^2 \cdot \frac{E_s}{4N_0} + d_e^2(\mathbf{x},\mathbf{x}') \cdot \frac{E_s}{4N_0} - \Delta_f^2 \cdot \frac{E_s}{4N_0}}\right)$$

$$\le \frac{1}{2} \cdot \sum_{\mathbf{x}' \in \Gamma} \operatorname{erfc}\left(\sqrt{\Delta_f^2 \cdot \frac{E_s}{4N_0}}\right) \cdot \exp\left(\Delta_f^2 \cdot \frac{E_s}{4N_0}\right) \cdot \exp\left(-d_e^2(\mathbf{x},\mathbf{x}') \cdot \frac{E_s}{4N_0}\right)$$

$$\le \frac{1}{2} \cdot \operatorname{erfc}\left(\sqrt{\Delta_f^2 \cdot \frac{E_s}{4N_0}}\right) \cdot e^{\Delta_f^2 \frac{E_s}{4N_0}} \cdot \sum_{\mathbf{x}' \in \Gamma} \exp\left(-d_e^2(\mathbf{x},\mathbf{x}') \cdot \frac{E_s}{4N_0}\right)$$

- Only the last term depends on the Euclidian distance between $\mathbf{x}$ and $\mathbf{x}'$
  → application of distance spectrum is again possible

Analytical Approximation of Bit Error Probability

- Total error probability:

$$P_e = \sum_{\mathbf{x}\in\Gamma} P(\text{Decoding error}, \mathbf{x}) = \sum_{\mathbf{x}\in\Gamma} P(\mathbf{x}) \cdot P(\text{Decoding error} \mid \mathbf{x})$$
$$= \sum_{\mathbf{x}\in\Gamma} P(\mathbf{x}) \cdot P_e(\mathbf{x})$$
$$\leq \frac{1}{2} \cdot \text{erfc}\left(\sqrt{\Delta_f^2 \cdot \frac{E_s}{4N_0}}\right) \cdot e^{\Delta_f^2 \cdot \frac{E_s}{4N_0}} \cdot \sum_{\mathbf{x}\in\Gamma} P(\mathbf{x}) \cdot \sum_{\mathbf{x}'\in\Gamma} \exp\left(-d_e^2(\mathbf{x},\mathbf{x}') \cdot \frac{E_s}{4N_0}\right)$$

- IOWEF of TCM encoder:

$$T(D,W) = \sum_{\mathbf{x}\in\Gamma} P(\mathbf{x}) \cdot \sum_{\mathbf{x}'\in\Gamma} D^{d_e^2(\mathbf{x},\mathbf{x}')} \cdot W^{w(\mathbf{x},\mathbf{x}')}$$

  - Considers difference of all sequences and their probability $P(\mathbf{x})$
  - $w(\mathbf{x},\mathbf{x}')$ denotes numbers of bit errors for $\mathbf{x} \rightarrow \mathbf{x}'$ $\quad w(\mathbf{x},\mathbf{x}') = d_H(\mathbf{u},\mathbf{u}')$

Analytical Approximation of Bit Error Probability

- Total error probability using distance spectrum

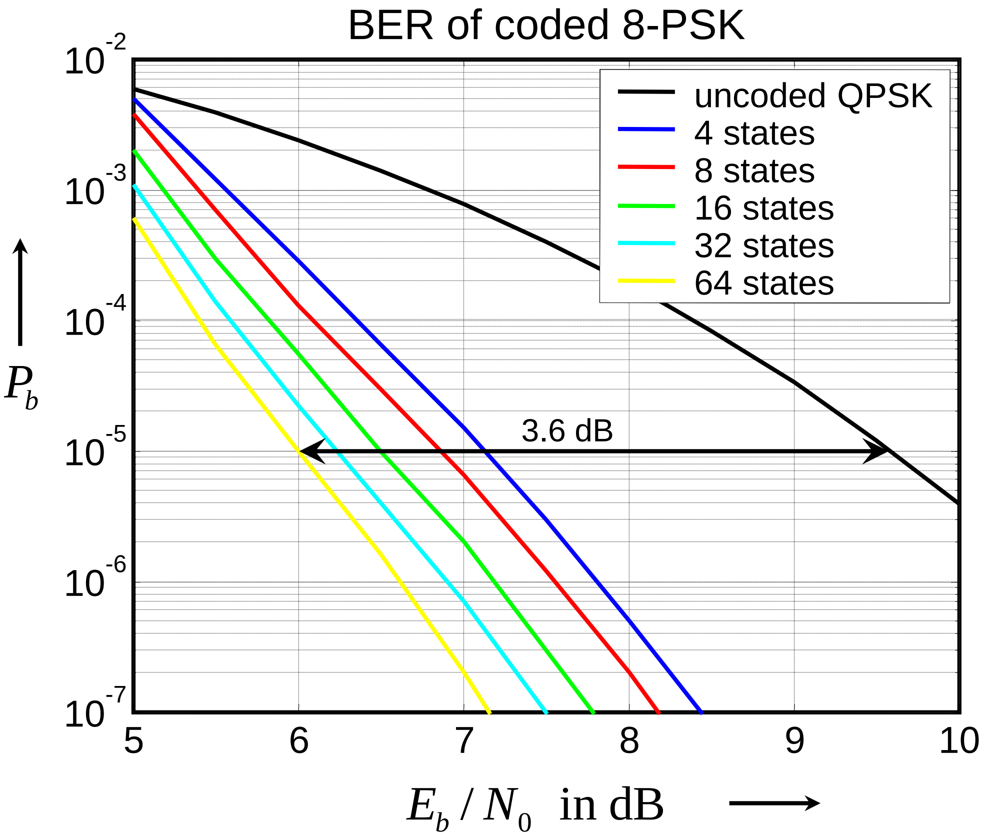
$$P_e \le \frac{1}{2}\cdot \mathrm{erfc}\left(\sqrt{\Delta_f^2 \cdot \frac{E_s}{4N_0}}\right)\cdot e^{\Delta_f^2 \cdot \frac{E_s}{4N_0}} \cdot T\left(D = e^{-\frac{E_s}{4N_0}}, W = 1\right)$$

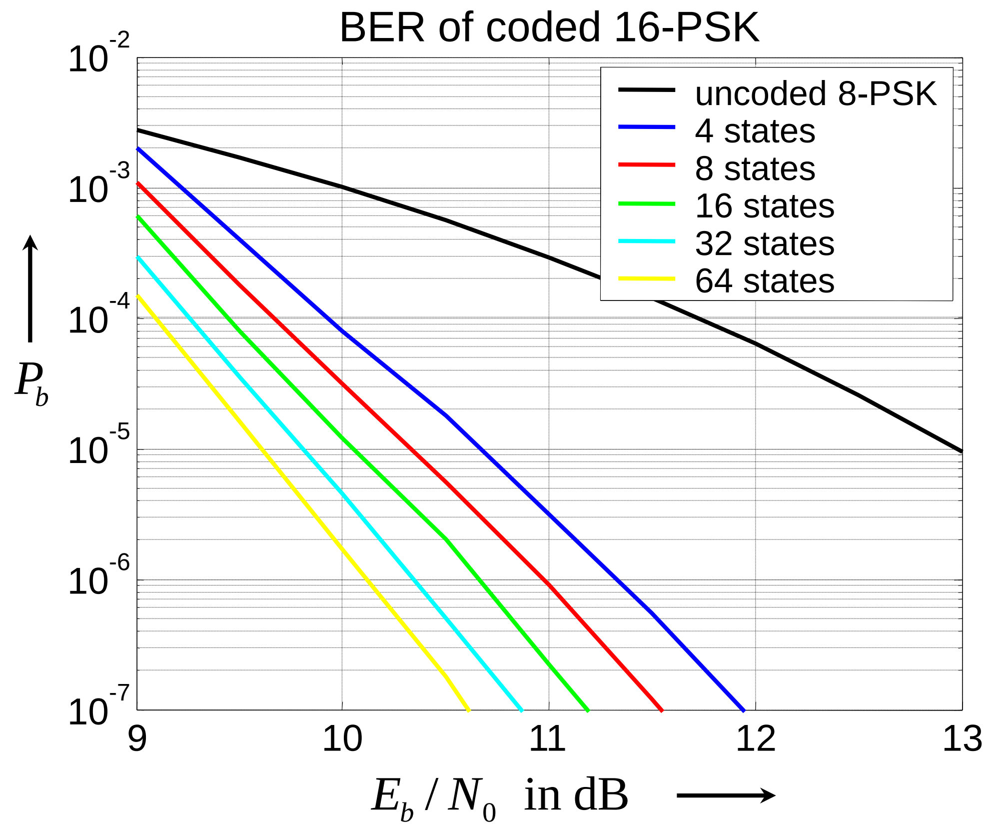
- Bit Error Probability
  - Number of differing information bits $w(\mathbf{x},\mathbf{x}')$ between $\mathbf{x}$ and $\mathbf{x}'$ has to be considered

$$P_b \le \frac{1}{2}\cdot \mathrm{erfc}\left(\sqrt{\Delta_f^2 \cdot \frac{E_s}{4N_0}}\right)\cdot e^{\Delta_f^2 \cdot \frac{E_s}{4N_0}} \cdot \sum_{\mathbf{x}\in\Gamma} P(\mathbf{x}) \cdot \sum_{\mathbf{x}'\in\Gamma} \frac{w(\mathbf{x},\mathbf{x}')}{m} \cdot \exp\left(-d_e^2(\mathbf{x},\mathbf{x}')\frac{E_s}{4N_0}\right)$$

$$= \frac{1}{2}\cdot \mathrm{erfc}\left(\sqrt{\Delta_f^2 \cdot \frac{E_s}{4N_0}}\right)\cdot e^{\Delta_f^2 \cdot \frac{E_s}{4N_0}} \cdot \frac{1}{m} \cdot \left.\frac{\partial T(D,W)}{\partial W}\right|_{D=e^{-\frac{E_s}{4N_0}},\, W=1}$$

Analytical Approximation of Bit Error Probability



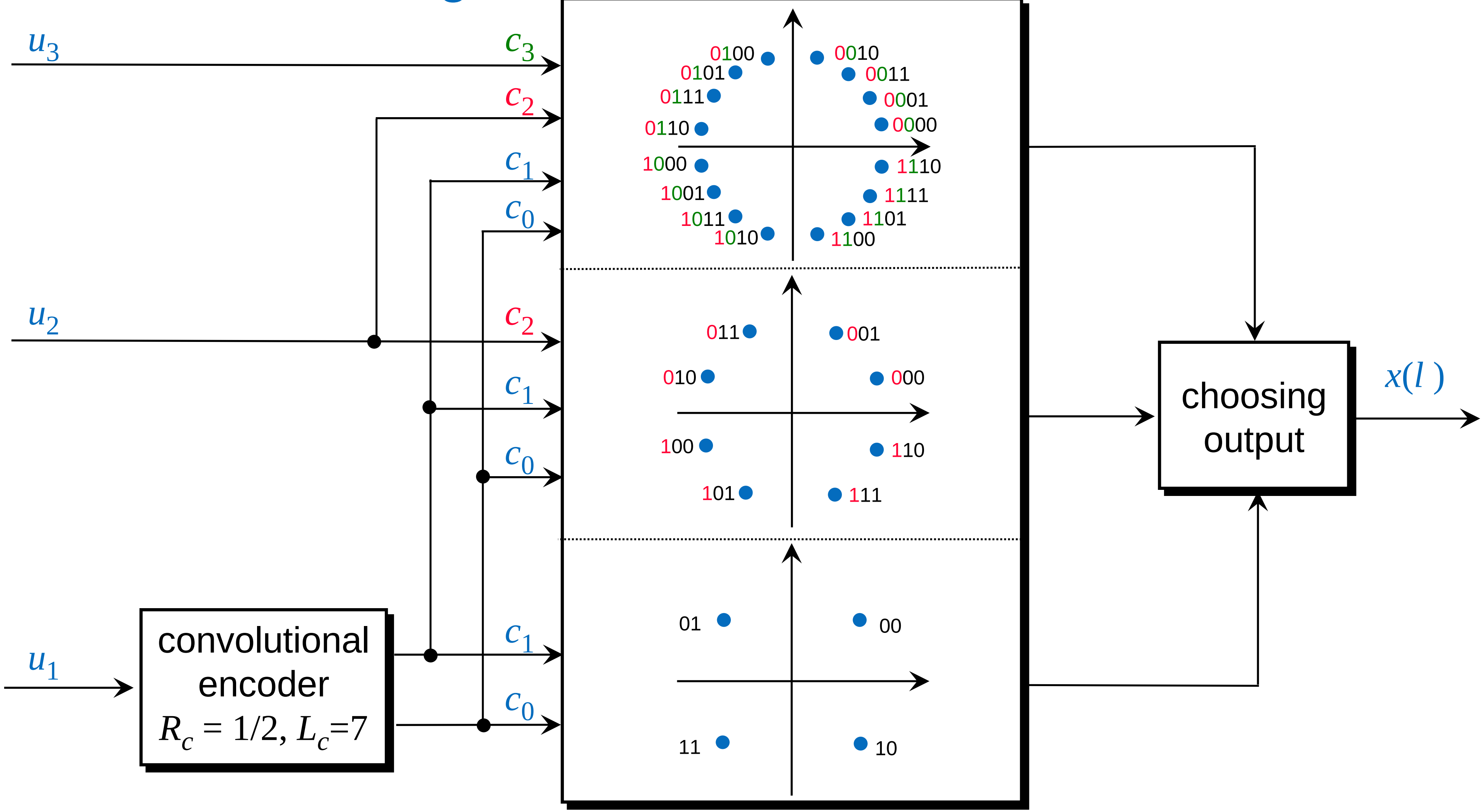


- Performance improves with increasing number of states → decoding effort grows exponentially
- Gain of 3.6 dB for 64 states is still 3.4 dB lower than promised by capacity
- Although theoretical gains are not achieved, strong performance improvements are obvious

Pragmatic Approach by Viterbi

- Modern communication systems require adjustment with respect to time variant channel properties and adaptivity with respect to requested data rates
  → flexibility and adaptivity are required
- Drawback for optimal codes of Ungerböck's
  - Switching between constellations with different spectral efficiencies η depending on the requirements is necessary
  - TCM was optimized for each η and requires different shift register structures and decoder (Viterbi)
- **Pragmatic Approach by Viterbi** (in general not optimal)
  - Conventional half-rate NSC code with $L_c=7$ in combination with different alphabets
  - η=1 bit/s/Hz → QPSK: u_1 is encoded → (c_1, c_0)
  - η=2 bit/s/Hz → 8-PSK: uncoded bit u_2 selects upper/lower signal set
  - η=3 bit/s/Hz → 16-PSK: uncoded bits (u_3, u_2) select quadrant

Pragmatic Approach by Viterbi



Pragmatic Approach by Viterbi

- η=1 bit/s/Hz → QPSK: u_1 is encoded → (c_1, c_0)
- η=2 bit/s/Hz → 8-PSK
  - u_1 is encoded → (c_1, c_0) → same encoder / decoder structure as η=1 bit/s/Hz
  - uncoded bit u_2 selects upper signal set (000,001,011,010) or lower signal set (100,101,111,110)
  - Code bits (c_1, c_0) determine symbol with signal set
  - 2 parallel branches to and from each state in Trellis diagram
- η=3 bit/s/Hz → 16-PSK
  - uncoded bits (u_3, u_2) select quadrant
  - Code bits (c_1, c_0) determine symbol with signal set → 4 parallel branches
- Very flexible structure, as varying spectral efficiency effects only number of uncoded bits but not encoder / decoder structure
  - Only small performance drawback in comparison to optimum TCM codes, e.g., pragmatic code for 8-PSK results in loss of 0,4 dB at $P_b=10^{-5}$

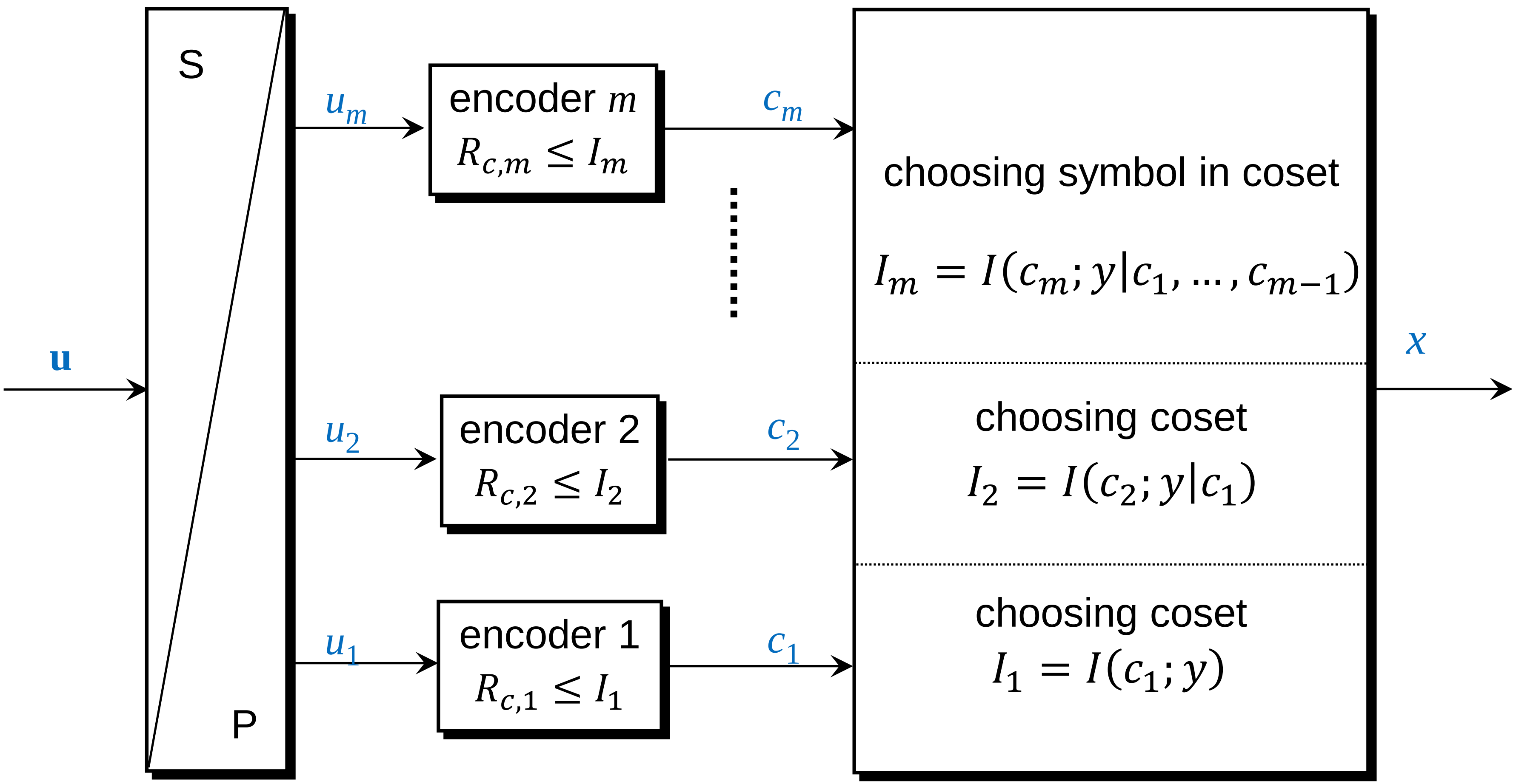
Multilevel Codes by Imai: Insights from Information Theory

- Bijective mapping of $m = \log_2(M)$ coded bits $c_1, \ldots, c_m$ onto symbol x
- Chain rule of mutual information

$$I(x;y) = I(c_1, \ldots, c_m; y) = \sum_{\mu=1}^{m} I(c_\mu; y \mid c_1, \ldots, c_{\mu-1}) = \sum_{\mu=1}^{m} I_\mu$$

- Interpretation:
  - Successive decoding of bits is optimal (reaches capacity)
    → Independent encoding of bit-levels
  - Already decoded bits have to be provided to successive decoders as a priori information
  - Order of detection can be arbitrary, but determines bit-level capacities
  - Each decoding stage has to be error-free
    → capacity achieving codes in each bit-level required

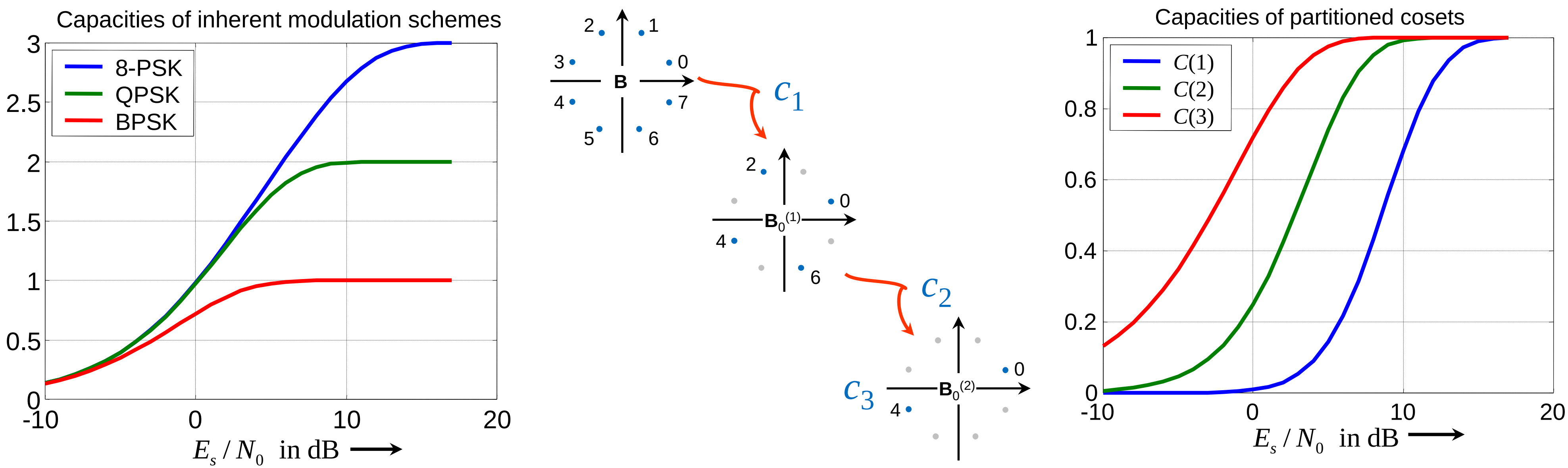
Multilevel Codes by Imai: Structure of Encoder



Multilevel Codes by Imai: Structure of Encoder

- Especially for fading channels the application of Imai's Multilevel codes (MLC) results in performance improvements
- Encoder structure:
  - No strict differentiation between coded and uncoded bits
  - Serial/Parallel conversion of information sequence
  - Different encoder (L_c, R_c) for each S/P output (**branch**), e.g. $R_c = 1$
- Possible strategy for encoder design
  - Separation of signal space by Ungerböck's Set-Partitioning
  - Due to increased Euclidian distance within each subset, the **equivalent channel capacity** (**bit-level capacity**) of each partitioning step increases
  - Each branch of the MLC decides about one partitioning step
  - Choose the code rate of each branch according to the **equivalent channel capacity**
  - Due to increased distance, the capacity increases with each partitioning step
    → i.e. codes become weaker with partitioning steps
    → e.g. uncoded bits in last partitioning steps

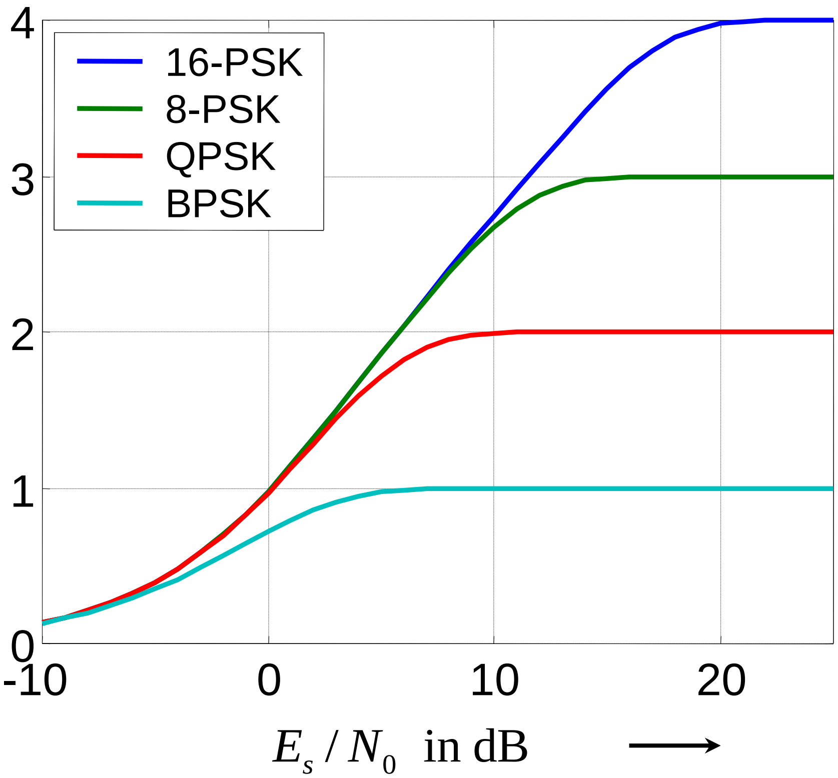
Multilevel Codes by Imai: Optimizing the Encoder for 8-PSK



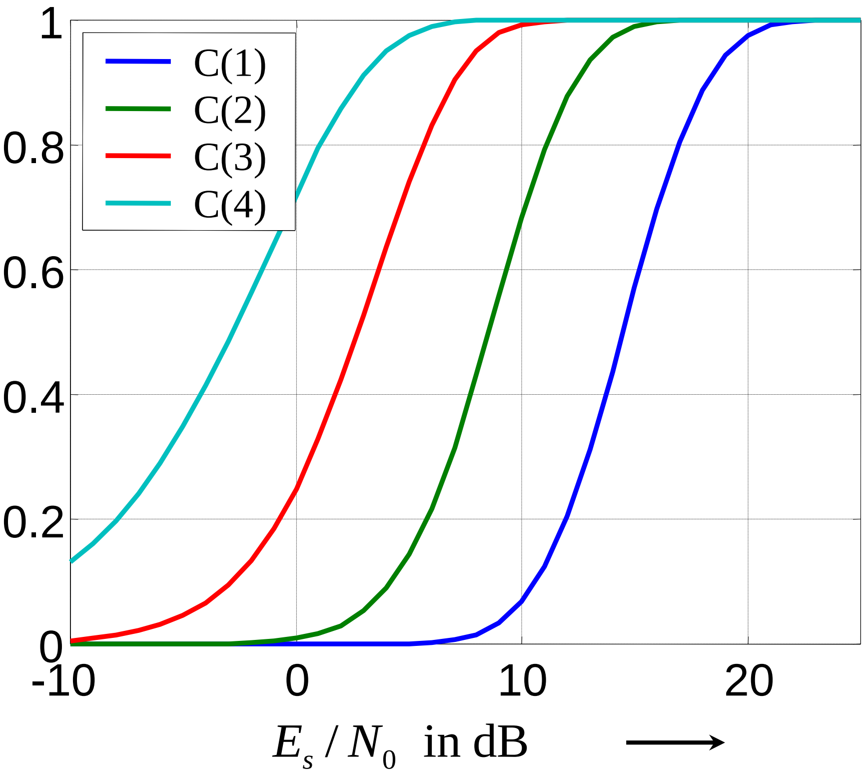
- 1st partitioning step generates 2 QPSK constellation each with capacity C(QPSK)
  - Code bit c_1 selects partitioning sets (not symbol within partitioning set)
  - Capacity of 1st step: $I_1 = C(1) = C(\text{8–PSK}) - C(\text{QPSK})$
- 2nd partitioning step generates 4 BPSK constellations each with capacity C(BPSK)
  - Capacity of 2nd step: $I_2 = C(2) = C(\text{QPSK}) - C(\text{BPSK})$
- With $I_3 = C(3) = C(\text{BPSK})$ we get $C(1) + C(2) + C(3) = C(8 - PSK)$

Multilevel Codes by Imai: Optimizing the Encoder for 16-PSK

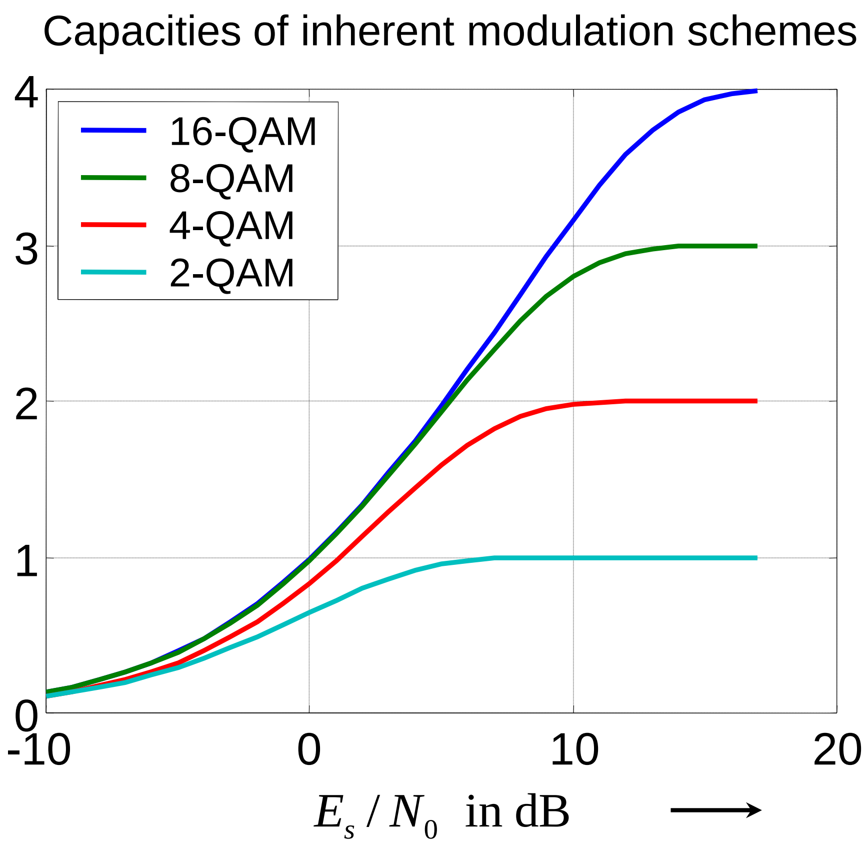
Capacities of inherent modulation schemes

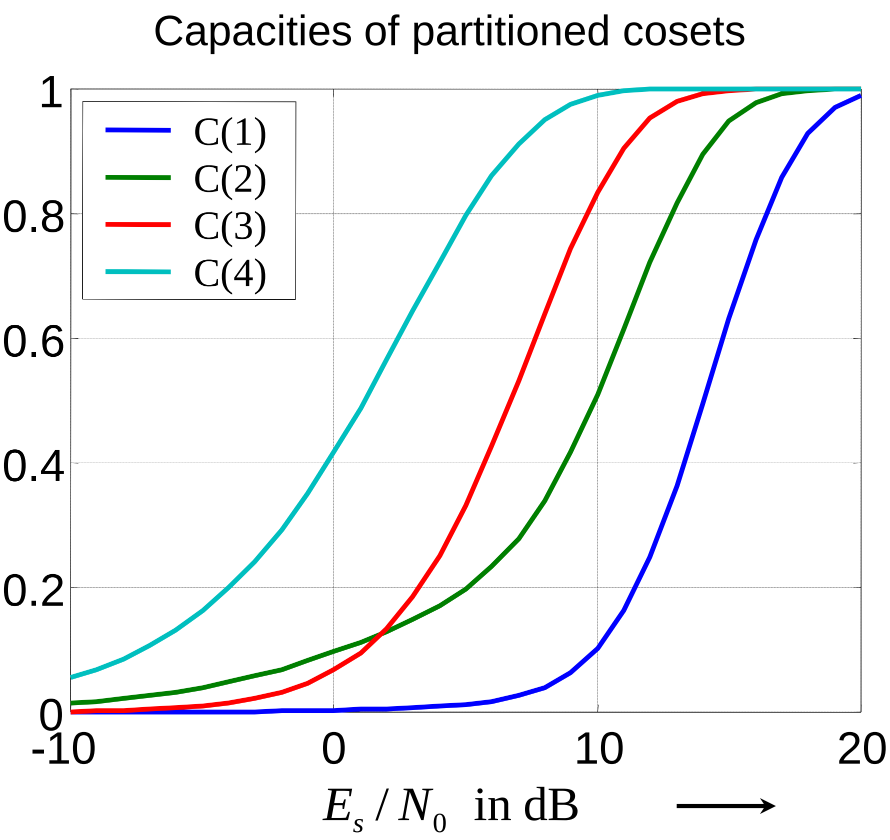


Capacities of partitioned cosets

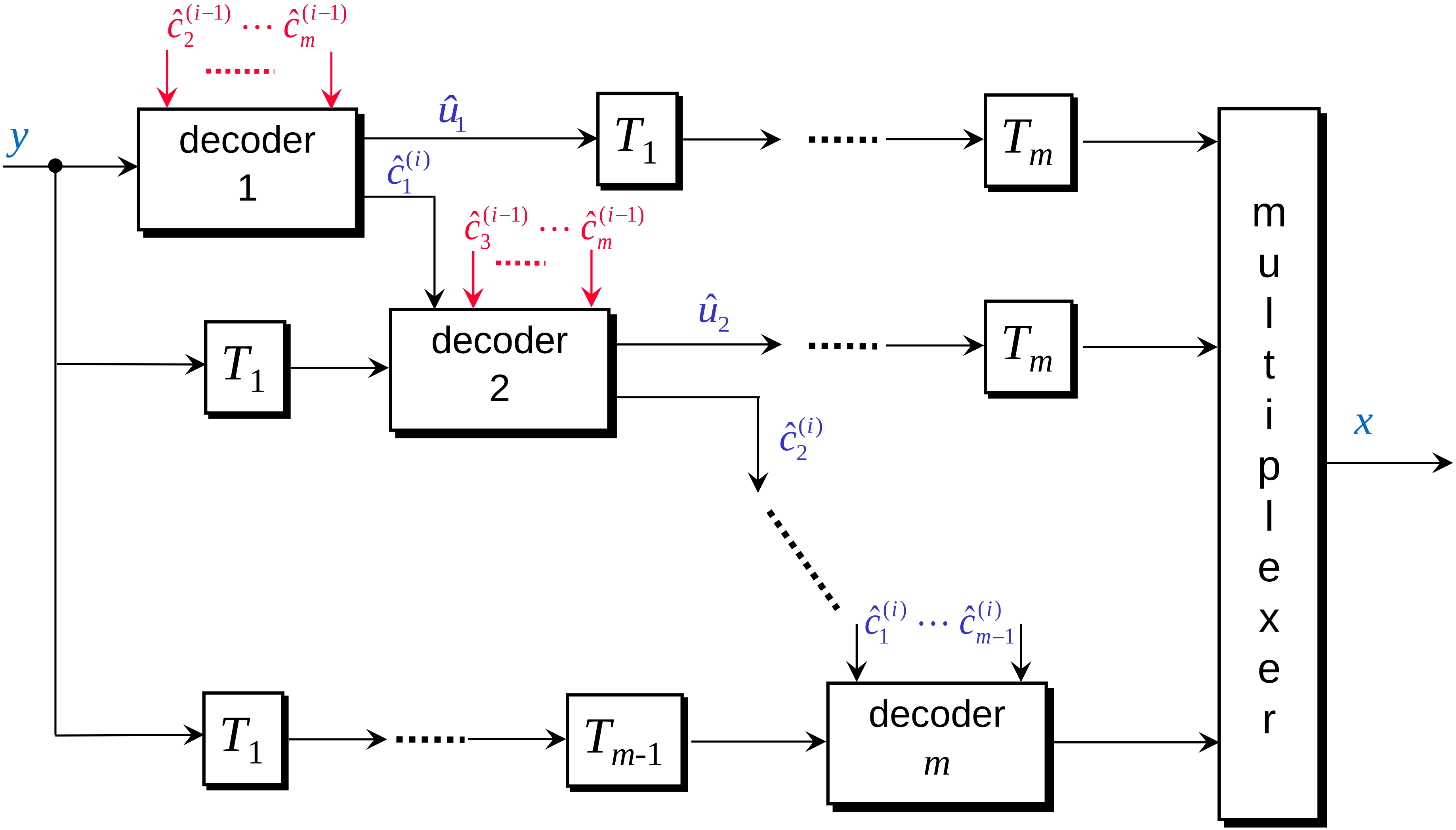


Multilevel Codes by Imai: Optimizing the Encoder for 16-QAM





Multilevel Codes by Imai: Iterative Decoding



TCM for telephone modems   Standards from ITU (CCITT)

- **V.26**: 1962 developed, 1968 standardized, uncoded 4-PSK, 2.4 kbit/s or 1.2 kbaud, fixed analog equalizer (designed for average channel profile)
- **V.27**: 1967, uncoded 8-PSK, 4.8 kbit/s or 1.6 kbaud, dispersive channel due to higher bandwidth $\rightarrow$ system becomes more sensitive, adaptive analog equalizer
- **V.29**: uncoded 16-QAM, 9.6 kbit/s or 2.4 kbaud, system becomes even more sensitive $\rightarrow$ digital equalizer at symbol rate 1/T

Since then, channel coding could be applied because technology becomes able to handle the problem of relative complex decoding algorithms for TCM.

- **V.32**: 1981, 32-QAM TCM (rotational invariant by Wei), 14.4 kbit/s or 3.6 kbaud, digital fractional tap spacing equalizer (over sampling)
- **V.33**: 128-QAM TCM, 14.4 kbit/s or 2.4 kbaud, 64-QAM TCM, 12 kbit/s or 2.4 kbaud
- **V.34**: 960-QAM TCM, adaptation to channel (channel estimation required), code rates R_c=2/3 (16 states), R_c=3/4 (32 states), R_c=4/5 (64 states), B = 3.2 kHz, 2.4 kbit/s ... 28.8 kbit/s

Data Rates of Communication Systems over Copper Telephone Lines

Transmission System	Bandwidth	Data rate
Analog telephone (POTS)	300 Hz – 3.4 kHz	up to 56 kbit/s (typically 4,5 kByte/s – 5 kByte/s)
ISDN	0 Hz – 120 kHz	2 · 64 kBit/s data channel + 16 kBit/s control channel
ADSL (ADSL-over-ISDN, Annex B)	U: 138 kHz – 276 kHz D: 276 kHz – 1.1 MHz	Upstream: 1 Mbit/s Downstream: up to 10 Mbit/s,
ADSL2+ (ADSL-over-ISDN)	U: 138 kHz – 276 kHz D: 276 kHz – 2.2 MHz	Upstream: 1 Mbit/s Downstream: up to 24 Mbit/s,

- Actual data rate depends strongly on distance to telephone switch